Hidden Points and Vertices in Classes of Polygons

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Abstract

The hidden point problem, is to arrange as many as possible points in a given polygon, such that any two of them are invisible to each other. When these points must be positioned on the vertices of the polygon, it is called the hidden vertex problem. Since they were proposed, many efforts have been made to capture their hardness, design approximation algorithms, and solve them efficiently in certain type of polygons. In this paper, these endeavors are continued, and our results are highlighted in a $\frac{2}{3}$-approximation algorithm of the maximum hidden vertex set in a pseudotriangle, and an exact algorithm for maximum hidden point set in a terrain or fan-shaped polygon. Concurrently, we also show that the decision problem of hidden point problem is in $\exists\mathbb{R}$.

Besides, the hidden points and hidden vertices are also closely related to other topics, including visibility graph, k-convexity, and convex covering. In this paper, we introduce novel combinatorial and geometric structures such as convex/reflex chains, set system of visible areas, and the continuous visibility graph, entitling us to establish original arguments and fresh insights in these realms.

1. Introduction

Two points in a polygon are visible to each other (w.r.t. to the polygon) if the segment connecting them lies inside the polygon. Given a polygon in the plane, a hidden point set is a set of points in the polygon such that any two of them are not visible to each other. If these points have further been placed on the vertices of the polygon, it is called a hidden vertex set. To find the largest hidden point set and hidden vertex set in a polygon, are the primary problems which this paper has the most keen interests on.

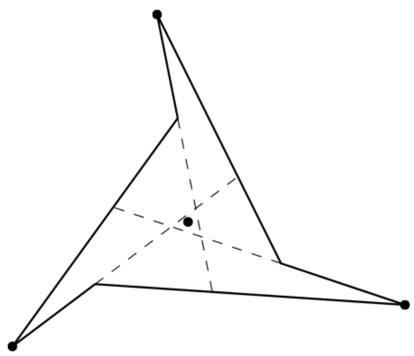

Figure 1: A hidden point set

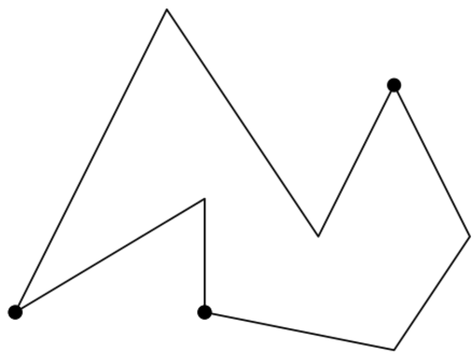

Figure 2: A hidden vertex set

Upon they were proposed in [7], they were immediately proven to be NP-hard. In terms of approximation, [4] showed that the hidden vertex problem is APX-hard, even if the polygon does not have holes. Meanwhile, [4] also proved that there exists $\epsilon > 0$ such that the maximum hidden point set can not be approximated by a polynomial algorithm within $1 + \epsilon$ unless P $=$ NP. This means a PTAS is not likely to exist, and they have been considered to be hard visibility problems so far.

Since then, many efforts have been committed to solve them efficiently in specific classes of polygons, for instance, weak visibility polygon [6], spiral polygon [5], and funnel polygon [2]. In addition to finding the exact optimal solution, approximation algorithms have also been paid a lot of attention. Given a simple polygon with n vertices, [1] proposed an $O(n^2)$ time algorithm to compute a 1/4-approximation of the maximum hidden vertex set, while [3] presented an $O(n^{2+o(1)})$ time algorithm that provides us with a $1/8$-approximation of the maximum hidden point set.

2. Our results

Our main contributions can be highlighted in the following aspects.

Theorem 1. *Let P be a fan-shaped polygon (or a terrain) with n vertices, the maximum hidden point set of P can be computed in $O(n^2)$.*

Lemma 2. *Let P be a pseudotriangle with n vertices, a $\frac{2}{3}$-approximation of the maximum hidden vertex set in P can be computed in $O(n^6)$.*

This approximation is achieved via dynamic programming, and can be applied analogously to polygons which have more than three convex vertices.

Corollary 3. *Let P be a simple polygon with n vertices, among which there are c convex vertices. A $\frac{2}{c}$-approximation of the maximum hidden vertex set in P can be computed in $O(n^6)$.*

Theorem 4. *Let P be a staircase polygon with n vertices, the maximum hidden point set and maximum hidden vertex set both can be computed in $O(n)$.*

Apart from proposing efficient algorithms, we also established theoretical arguments via the geometric and combinatorial structures introduced in our paper. Some notable ones are presented as follow.

Lemma 5. *Let P be a simple polygon with r reflex vertices, and $\mathcal{F} = \{Vis(u) | u \in P\}$ be the family of visible areas of P. The VC-dimension of the set system $(P, \mathcal{F})$ is bounded by $O(\log r)$.*

Lemma 6. *Let P be a simple polygon and k be a positive integer. To decide whether P admits a hidden point set of size k is in $\exists\mathbb{R}$.*

Lemma 6 shows the $\exists\mathbb{R}$-membership of the decision problem of the hidden point set. However, its completeness is negatively suggested from our point of view.

Lemma 7. *Let P be simple polygon. Let $hp(P)$ and $cc(P)$ denoted the size of maximum hidden point set in P and minimum convex covering of P. If $hp(P) \leq 2$, then $cc(P) \leq 3$.*

It has been proven in [3] that $cc(P) \leq 8hp(P)$ for any simple polygon P. However, the largest possible gap between them remains open, and is conjectured to be much less than 8.

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