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Hidden Points and Hidden Vertices: Hard Visibility Problems in Classes of Polygons

Master Thesis

Yuda Fan

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Advisor: Prof. Dr. Bernd Gärtner

Department of Computer Science, ETH Zürich

幾何学の本質は、性質だけではありません要素 (必ずしも多様体である必要はない) だけでなく、方法でもこれらの要素が構成されています。

私たちは、幾何学は自然法則つであると信じています、自然自体は面白くて美しいです。

The essential point in geometry does not only lie in the properties of the elements (which are not necessarily manifolds), but also in the ways these elements are composed.

We believe that geometry reveals laws of the nature, and nature itself is interesting and beautiful.

出現の根本原理 【イコリア：巨獣の棲処】

Abstract

The hidden point problem, is to arrange as many as possible points in a given polygon, such that any two of them are invisible to each other. When these points must be positioned on the vertices of the polygon, it is called the hidden vertex problem. Both of them have been proved to be hard visibility problems, in terms of either decision or approximation.

Since they were proposed, many efforts have been made to capture their hardness, devise approximation algorithms, and solve them efficiently in certain type of polygons. In our thesis, these endeavors are continued, and our results are highlighted in a $\frac{2}{3}$-approximation algorithm of the maximum hidden vertex set in a pseudotriangle, and an exact algorithm for maximum hidden point set in a terrain or fan-shaped polygon. Notably, both of them run in polynomial time, and it is the first time they are resolved or provided with a non-trivial approximation. Concurrently, we also show that the decision problem of hidden points is in $\exists\mathbb{R}$.

Besides, the hidden points and hidden vertices are also closely related to other topics, including visibility graph, k-convexity, and convex covering. In this thesis, we introduce novel combinatorial and geometric structures such as convex/reflex chains, set system of visible areas, and the continuous visibility graph, entitling us to establish original arguments and fresh insights in these realms.

Contents

$[n]$	the set $\{1, 2, \cdots, n\}$
2^X	the power set of X
G	graph
$\overline{G}$	the complement graph of G
$G[X]$	induced subgraph of G on vertex set X
$V(G)$	vertex set of G
$E(G)$	edge set of G
$(u, v) \in E(G)$	the edge between vertex u and v
$N(u)$	neighbour of vertex u
$\alpha(G)$	independence number of G
$\chi(G)$	chromatic number of G
$\omega(G)$	clique number of G
$\kappa(G)$	clique cover number of G
P	polygon
∂P	boundary of P
$P(u, v)$	polygonal chain of ∂P from point u to v
$\mathrm{HP}(P)$	hidden point number of P
$\mathrm{HV}(P)$	hidden vertex number of P
$\textsc{Cover}(P)$	convex covering number of P
$\textsc{Partition}(P)$	convex partition number of P
$VG(P)$	visibility graph of P
$CVG(P)$	continuous visibility graph of P
$\overline{uv}$	the segment connecting the point u and v
$\overrightarrow{uv}$	the ray starting from u and oriented to v
X	point set in the plane
$int(X)$	interior of X
$relint(X)$	relative interior of X
$conv(X)$	convex hull of X
$B(u, r)$	ball centering at point u with radius r

Chapter 1

Introduction

Given a polygon in the plane, a hidden point set is a set of points in the polygon such that any two of them are not visible to each other. If these points have further been placed on the vertices of the polygon, it is also called a hidden vertex set. To find the largest possible hidden point set and hidden vertex set in a specific polygon, are the primary problems which this thesis has the most keen interests on.

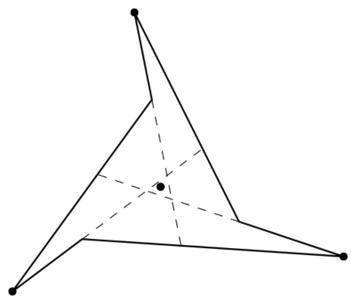


Figure 1.1: A hidden point set

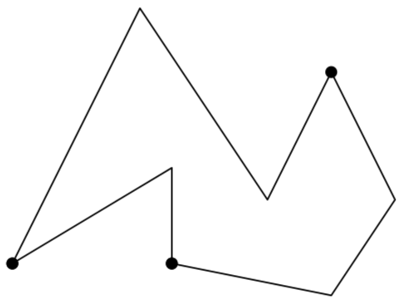


Figure 1.2: A hidden vertex set

Hidden Points and Hidden Vertices. Upon they were proposed in [43], they were immediately proven to be NP-hard. This means a polynomial time algorithm is not likely to exist, and they have been considered to be hard visibility problems so far. Since then, many efforts have been committed to solve them in specific class of polygons, for instance, weak visibility polygon [28], fan-shaped polygon [29], and funnel polygon [9].

In addition to finding the exact optimal solution, efficient approximation algorithms have also been paid a lot of attention. Given a simple polygon with n vertices, [7] proposed an $O(n^2)$ time algorithm to compute a 1/4-

approximation of the maximum hidden vertex set , while [10] presented an $O(n^{2+o(1)})$ time algorithm that provides us with a $1/8$-approximation of the maximum hidden point set. Both algorithms are based on the staged illumination technique [47], and are known as the first constant factor approximation for the corresponding problem.

Of course, these two problems are not isolated islands in the kingdom of computational geometry. Instead, through the study on them, the following topics have been shown intimately connected to them.

Visibility Graph. The visibility graph is an important combinatorial structure in computational geometry. In the visibility graph of a polygon, a pair of vertices are connected if they are visible to each other. Before the hidden point problem was proposed, the visibility graph had already demonstrated its importance by playing a critical role in many geometry algorithms, such as computing the geodesic path in presence of obstacles [35] and decomposing the two dimensional shape [42].

To compute the visibility graph of a polygon, [49] proposed an $O(n^2)$ algorithm, followed by [31] presenting an $O(m + n \log\log n)$ algorithm, where m is number of the edges in the visibility graph. With the linear time triangulation algorithm in [13], the one in [31] was further improved to $O(n + m)$.

Deciding whether the given graph is the visibility graph of a polygon or not, is referred to as the recognition problem, which is indeed another primary objective in the study of visibility graphs. [23] showed that it is in the complexity class $\exists\mathbb{R}$, but whether it belongs to NP or not has remained open so far. There are many attempts to propose necessary conditions for the visibility graph, including [26, 27, 1], but none of them has been proven to be sufficient up to now. Though this problem is considered to be hard in general case, the visibility graph of some special classes of polygon still can be recognized efficiently, likewise spiral polygon [24] and funnel polygon [14].

NP Optimization Problem. NP optimization problem class is a class of mathematical optimization problems, of which the recognition of the instance, feasibility of the solution, and the objective function can always be computed in polynomial time. Since it was proposed in [16], many hard optimization problems have been located in its compendium. A notable result closely related to our thesis it that [19] showed that the hidden vertex problem is APX-hard, even if the polygon does not have holes. Meanwhile, [19] also proved that there exists $\varepsilon > 0$ such that the maximum hidden point set can not be approximated by a polynomial algorithm within $1 + \varepsilon$ unless $P = NP$.

Existential Theory of the Reals. The complexity class $\exists\mathbb{R}$ has been a cornerstone in computational geometry and algebraic geometry, which includes many famous geometry problems. For example, [32] showed that the deciding whether a graph is the intersection graphs of line segments or not is in $\exists\mathbb{R}$, and [40] proved that the recognition of unit distance graphs also belongs to $\exists\mathbb{R}$. Besides, [5] shows that the well-established art gallery problem is $\exists\mathbb{R}$-complete.

Convex Covering. A convex covering of a polygon is a set of convex pieces, of which the union is exactly the same as the polygon. To find the convex covering with least possible number of pieces, called the convex covering problem, is closely related to the hidden point problem. The first notable result in its history was [36] showed that the convex covering with Steiner points is decidable. Later, [17] proved that it is NP-hard to decide whether a polygon can be covered by certain number of convex pieces, even if the polygon is simple. The first non-trivial approximation algorithm is given by [20], in which their algorithm has an $O(\log n)$ approximation ratio guarantee. When it comes to covering a simple polygon, [10] presented a 6-approximation algorithm, which is the best approximation ratio known so far.

Our Contributions. Our main contributions can be highlighted in the following aspects, in the order by which they are arranged in this thesis.

- We established a series of combinatorial proposition about maximum hidden point set, maximum hidden vertex set, and minimum convex covering. We illustrated how the number of holes guarantees the lower and upper bounds in Lemma 3.15 and Lemma 3.16. This problem is originated from the open problem workshop GWOP'23 and we partially answered it.
- We firstly discussed the structure of the visible area, by characterization of its Steiner points and windows. As the VC-dimension is an important metric to evaluate the complexity of a set system, we prove that the set system of the visible areas have finite VC-dimension, which is upper bounded by the logarithm of number of reflex vertices in Lemma 4.14.
- Whether irrational points are needed in the optimal solution or not is a critical point in many geometry problems. For example, [4] showed that sometimes they are necessary for the art gallery

problem. We proved that the irrational coordinates are unnecessary for hidden point set in Lemma 5.5, and investigate if the hidden point problem belongs to NPO in Lemma 5.12.

- Up to now, the upper bound on complexity of the hidden point problem has not been precisely captured. We proved that it belongs to the complexity class $\exists\mathbb{R}$ in Theorem 5.28.
- We introduced a generalized definition of convex chain and reflex chain. They used to be part of the boundary of the polygon, but they are also allowed to travel through the interior. We established their connection to the hidden point/vertex set and the clique in Lemma 6.7 and Lemma 6.11.
- The traditional visibility graph can not precisely characterize the hidden points and convex covering, and that is the reason why we introduce the continuous visibility graph. We demonstrate its utilization by connecting it to the hidden point set and convex covering. Meanwhile, we proved that the continuous visibility graph of a spiral polygon is chordal in Theorem 6.27.
- How large can the gap between maximum hidden point set and minimum convex covering be? To answer this, the polygon that only hosts two hidden points is a cute case to start with. We proved such polygon needs at most three convex pieces to cover in Lemma 7.19.
- We present a $\frac{2}{3}$-approximation algorithm for finding the maximum hidden vertex set of the pseudotriangle in Lemma 8.28, which runs in $O(n^2)$ time. Previously, the only known competitive ratio is $\frac{1}{2}$, which is indeed trivial.
- We present an $O(n^2)$ algorithm in Theorem 9.35, which efficiently solves the maximum hidden point set in the fan-shaped polygon and terrain.

Important Remarks. These are the important remarks that our reader should be aware throughout the thesis.

- The computational model we adopt here is the real RAM model [41], which means any allowed operation takes $O(1)$ time.
- We assume that geometric objects we discuss about are in an affine space, unless we specify the origin or the coordinates.
- We assume that the vertices of a polygon are in general position unless we specify a different assumption.

- The author himself is positive about P $\neq$ NP and NP $\neq \exists\mathbb{R}$, and there might be normative statements without assuming them.

Acknowledgement. This thesis serves as the diploma thesis at ETH Zürich, of which Prof. Dr. Bernd Gärtner is the supervisor. Thank him for his careful supervision, warm-hearted encouragement, and especially for proofreading the draft.

Chapter 2

Polygon and Visibility Graph

Polygons are the cornerstone and the fundamental objects in computational geometry. When it comes to the visibility problems concerning the polygon, the visibility graph appears as a valuable combinatorial structure. In this chapter we introduce the basic definitions and background knowledge about them, especially various classes of polygons.

2.1 Classes of Polygons

2.1.1 Definition of Polygon

Given $n \geq 3$ points in the plane with general position, there are many different ways to connect them with straight-line segments into a cycle, and thereby resulting in a polygon. Through this process, we can give a proper definition for the polygon on n points.

Definition 2.1 *Let $\{p_0, p_1, \cdots, p_{n-1}\}$ be n points in the plane, in which only p_0 and p_{n-1} are allowed to be identical to each other. The polygonal chain $C = (p_0, \cdots, p_{n-1})$ is an ordered set of edges, which is composed by the segment $e_i = (p_i, p_{i+1}), i \in [0, n-2]$ consecutively.*

Definition 2.2 *Let $C = (p_0, \cdots, p_{n-1})$ be a polygonal chain, it is called a closed polygonal chain if $p_0 = p_{n-1}$.*

Definition 2.3 *Let $X = \{x_1, x_2, \cdots, x_n\}$ be n points in the plane with general position, the polygon P denote the closed region enclosed [1] by the closed polygonal chain $\partial P = C = (x_1, x_2, \cdots, x_n, x_1)$.*

[1] A point is said to be enclosed by a closed curve Γ if either it is in Γ or any connected path from it to the point of infinity crosses Γ odd times

Meanwhile, we characterize P by the pair (V, E), where $V = X$ is the vertices of P, and $E = \{(x_i, x_{i+1})|i \in [n]\}$ is the edges of P.

In this thesis, without specification, we always discuss polygons with vertices are in general position (any three vertices are not collinear). For simplicity, we also write P as $(x_1, x_2, \cdots, x_n)$, which means P is a polygon on vertices $\{x_1, x_2, \cdots, x_n\}$ and the edges of P is given by the closed polygonal chain $C = (x_1, x_2, \cdots, x_n, x_1)$. Further, we always assume that x_i is indexed in counter-clockwise order[2]. Meanwhile, when we write $e_i = (x_i, x_{i+1})$ as an edge of a polygon or a polygonal chain, we actually refer to the segment $\overline{x_i x_{i+1}}$.

2.1.2 Simple Polygon

Definition 2.4 *Let a, b be two segments in the plane, which are not collinear to each other, then segments a and b are called properly intersect with each other if*

$$relint(a) \cap relint(b) \neq \varnothing,$$

where $relint(x)$ is the relative interior of x.

Definition 2.5 *Let P be a polygon, P is called a simple polygon if and only if $\forall e_1, e_2 \in E(P), e_1 \neq e_2$, e_1 and e_2 intersect with each other only if they are consecutive on the boundary of P, in which case they only share a common endpoint.*

Let P be a simple polygon on X. Since $E(P)$ does not intersect with itself, the edges of P compose a planar simple curve. By the Jordan Curve Theorem [48], it divides the plane into two disjoint parts: the "interior" region inside P and the "exterior" region which includes the point at infinity. Denote them by $int(P)$ and $ext(P)$ respectively and the curve itself by ∂P. Accordingly, we have $\mathbf{R}^2 = \partial P \cup int(P) \cup ext(P)$.

Definition 2.6 *Let $C = (p_0, p_1, \cdots p_{n-1})$ be a polygonal chain without self-intersection and $u, v \in C$ be two distinct points on C. Suppose that $u \in (p_s, p_{s+1}), v \in (p_t, p_{t+1})$, and u appears before v in C.*

The subchain $C(u, v)$ is defined as

$$C(u, v) := \begin{cases} (u, v), & s = t, \\ (u, p_{s+1}, \cdots, p_t, v), & s < t. \end{cases}$$

[2] By counter-clockwise order, we are indicating that the interior of P lies one the left hand side of $\overrightarrow{x_i x_{i+1}}$

Definition 2.7 *Let $P = (p_0, p_1, \cdots p_{n-1})$ be a simple polygon, and $u, v \in \partial P$ be two distinct points in ∂. The polygonal chain $P(u, v)$ is defined as the subset of ∂P which routes from u to v in counter-clockwise order.*

Let P be a simple polygon and u, v be two distinct points on its boundary. There are exactly two polygonal chains through ∂P routed from u to v, one in clockwise order and another in counter-clockwise order, and the union of them covers ∂P. Thus, $\partial P = P(u, v) \cup P(v, u), \{u, v\} = P(u, v) \cap P(v, u)$.

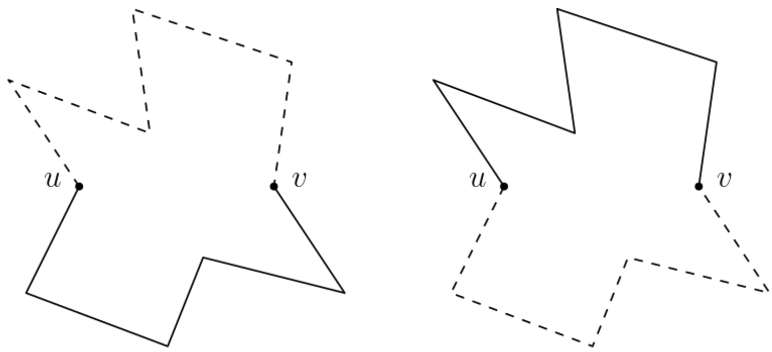


Figure 2.1: The left one indicates $P(u, v)$ and the right one indicates $P(v, u)$.

2.1.3 Polygon with Holes

Definition 2.8 *Let P be a simple polygon, a hole H in P is a simple polygon such that $H \subseteq int(P)$.*

Let P be simple polygon and $\mathcal{H} = \{H_1, H_2, \cdots, H_m\}$ be collection of holes such that

$$\forall i \in [m], \quad H_i \subseteq int(P),$$
$$\forall i \neq j, \quad H_i \cap H_j = \varnothing,$$

then the polygon Q with holes is defined as $Q := P \setminus \bigcup_{i=1}^{m} int(H_i)$.

Hence, the vertex set of Q is defined as $V(Q) := V(P) \cup V(H_1) \cup \cdots \cup V(H_m)$, and the edge set of Q is defined as $E(Q) := E(P) \cup E(H_1) \cup \cdots \cup E(H_m)$. Further, the polygon P is called the body of Q and $\mathcal{H}$ is called the holes of Q respectively.

2.1.4 Star-shaped polygon

Definition 2.9 *Let P be a simple polygon and p, q be two distinct points. p and q are called visible to each other if and only if $\overline{pq} \subseteq P$. We denote the visibility* *between points p and q with regard to polygon P by*

$$I_P(p,q) = \begin{cases} 1, & \overline{pq} \subseteq P, \\ 0, & otherwise. \end{cases}$$

To be specific, we allow the line of vision $\overline{pq}$ to remain unblocked even if it touches ∂P. For example, let (u, v) be an edge of P, then u and v are actually visible to each other.

Definition 2.10 *Let P be a simple polygon, P is called a star-shaped polygon if and only if*

$$\exists u, s.t. \forall v \in P, \overline{uv} \subseteq P.$$

Moreover, ker(P), denoting the kernel of P, is defined by

$$ker(P) := \{u | \forall v \in P, \overline{uv} \subseteq P\}.$$

Accordingly, P is a star-shaped polygon implies that $ker(P) \neq \emptyset$.

2.1.5 Fan-shaped Polygon

Definition 2.11 *Let P be a simple polygon and $u \in P$ be a vertex of P, v is called a convex vertex if its interior angle is less than π and a reflex vertex if its interior angle is greater than π.*

Definition 2.12 *Let P be a simple polygon, P is called a fan-shaped polygon if and only if there exists a convex vertex $u \in P$ such that $u \in ker(P)$. In this case, vertex u is called the "hub" of P,*

$$hub(P) := \{u | u \in ker(P) \land u \text{ is a convex vertex of } P\}.$$

It is clear that a fan-shaped polygon is also a star-shaped polygon since $ker(P) \supseteq hub(P) \neq \emptyset$.

Proposition 2.13 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a fan-shaped polygon and $p_0 \in hub(P)$, then P admits a fan triangulation [37] $T = \{(p_0, p_i, p_{i+1}) | \forall i \in [n-2]\}$, where all the triangles share the common vertex p_0.*

2.1.6 Spiral Polygon

Definition 2.14 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a simple polygon, where p_0 is a convex vertex. P is a spiral polygon if and only if all its reflex vertices are arranged consecutively around the boundary. In other words, there exist $0 < l \leq r < n$ such that p_i is a reflex vertices if and only if $l \leq i < r$.*

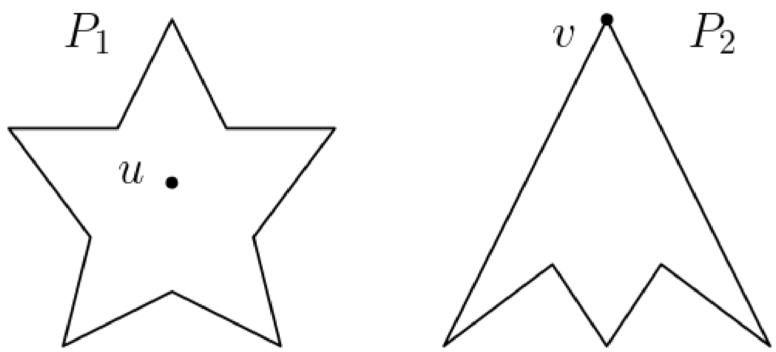


Figure 2.2: P_1 is a star-shaped polygon and $u \in ker(P_1)$, while P_2 is a fan-shaped polygon and $v \in hub(P_2)$.

2.1.7 Rectilinear Polygon

Definition 2.15 *Let P be simple polygon, P is a rectilinear polygon if the interior angle at each vertex is either $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$.*

Remark 2.16 *Let P be a rectilinear polygon and let $u = (x_u, y_u)$ be the X-Y coordinates of the vertex u. For convenience, we always assume that the edges of P are aligned to the axis. In other words, for any edge $e = (u, v)$ of P, we have either $x_u = x_v$ or $y_u = y_v$.*

2.1.8 Terrain

Definition 2.17 *A terrain T is a polygonal chain $T = (p_0, p_1, \cdots, p_{n-1})$ such that it is strictly x-monotone, which means that $\forall i \in [0, n-2], x_{p_i} < x_{p_{i+1}}$.*

There are two points which we consider very important here.

- The “terrain” in our context actually refers to the 1.5D terrain (which mean it is monotone in x-coordinates, but free in y-coordinates). Sometimes in other contexts, people prefer using “terrain” to refer to the 2.5D terrain.
- Since we are discussing the nature of geometric objects, we are actually dealing with the affine space, in which the origin and the coordinates are actually not specified. Therefore, let T be a polygonal chain, and if there exists a way to equip the affine space with an origin and $X - Y$ coordinate system such that T is x-monotone, we say that T is a terrain.

2.2 Visibility Graph

2.2.1 Visibility Graph of Simple Polygon and Terrain

Definition 2.18 *Let P be a simple polygon on n vertices, the visibility graph of P, denoted by $VG(P)$, is the graph on the vertices of P such that each pair of vertices are connected if and only if they are visible to each other.*

According to the remark about Definition 2.9, for any edge (u, v) of the polygon P, it is also an edge of the visibility graph $VG(P)$. Therefore, given that $P = (p_0, p_2, \cdots, p_{n-1})$ is a simple polygon, we have its boundary $(p_0, p_2, \cdots, p_{n-1}, p_0)$ is indeed a Hamiltonian cycle in $VG(P)$.

Definition 2.19 *Let P be a simple polygon, u be a vertex of P, and X be a set of vertices of P. We define $N(u)$ as the set of vertices, which is visible from u, $N(u) := \{v | v \in V(P), \overline{uv} \subseteq P\}$. We define $N(X)$ as the set of vertices, which is visible from any vertex in X, $N(X) := \bigcup_{u \in X} N(u)$.*

Actually, $N(u)$ is the set of vertices which is dominated by vertex u in the visibility graph.

Similar to the simple polygon, we can also define the visibility graph for a terrain as follow.

Definition 2.20 *Let T be a terrain, and u, v be two points such that $u \in T$ and $v \in T$. Then u and v is considered to visible to each other if $\overline{uv}$ lies completely above T.*

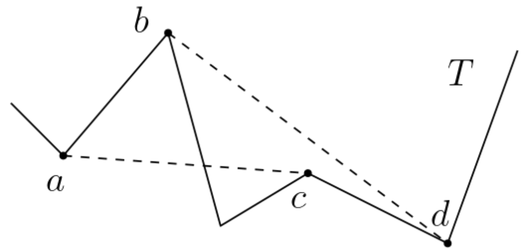


Figure 2.3: This figure illustrates the visibility on the terrain T. Here, (a, c) is invisible to each other, while (b, d) is visible to each other.

To be specific, $\overline{uv}$ is said to be completely above T if and only if there is no terrain vertex $r \in V(T)$ such that r is strictly above the segment $\overline{uv}$.

Definition 2.21 *Let T be a terrain, its visibility graph $VG(T)$ is defined as $VG(T) = (V, E)$, where V is the vertices of T and E is the edges connecting the pair of vertices which are visible to each other.*

Proposition 2.22 *Let P be a simple polygon on n vertices, $VG(P)$ can be computed in time $O(n \log n + e)$, where e is the number of edges in $VG(P)$.*

Proof This is proved in [31]. □

2.2.2 Recognition and Reconstruction of Visibility Graph

Apart from computing the visibility graph, there are two problems people shares keen interests on. One is the recognition of the visibility graph. In other words, for a specific given graph G, decide whether it is the visibility graph of some simple polygon P or not. However, notice that to be the visibility graph of a simple polygon, G is necessary to be Hamiltonian at first, which is already NP-complete to decide. Therefore, the recognition problem is always defined with a Hamiltonian cycle specified in G as follow.

Definition 2.23 *Let $G = (V, E)$ be a graph and C be a Hamiltonian cycle in G, the recognition problem is to decide whether there exists a simple polygon P such that $VG(P)$ is isomorphic to G and the boundary of P corresponds to C.*

Another one is the reconstruction of the simple polygon, given its visibility graph. Similar to the prior one, we could define it as follow.

Definition 2.24 *Let G be the visibility graph of the simple polygon P and C be a Hamiltonian cycle in G. The reconstruction problem is to figure out the arrangement of the vertices of P in the plane such that $VG(P)$ is isomorphic to G and the boundary of P corresponds to C.*

Sometimes, it is also called the realization of the visibility graph. A graph G is said to be realizable if such simple polygon exists.

2.2.3 Necessary Conditions for Visibility Graph

We introduce the first two necessary conditions adopted in [26], which is widely referred after.

Lemma 2.25 [NC1] *Let $G = (p_0, p_1, \cdots, p_{n-1})$ be the visibility graph of a simple polygon and $C = (q_0, q_1, \cdots, q_{k-1}), k \geq 3$ be an ordered cycle in G, then C has at least $k - 3$ chords.*

Lemma 2.26 [NC2] *Let $G = (p_0, p_1, \cdots, p_{n-1})$ be the visibility graph of a simple polygon P, and (p_i, p_j) be a pair of invisible vertices in G, then there exists a blocking vertex p_k with regard to (p_i, p_j).*

A blocking vertex p_k is the vertex such that the polygonal chain routed from p_i to p_j via the vertex p_k, is partitioned into two independent chains C_1 and C_2 such that $\forall u \in C_1, v \in C_2$, u and v are invisible to each other unless $u = v = p_k$.

In essence, the recognition problem itself is to find necessary and sufficient conditions for visibility graph that can be efficiently verified. So far, the hardness of this problem remains open. But still, there are classes of polygons whose visibility graphs can be recognized in polynomial time, for instance, spiral polygons [24] and funnel polygons [14].

In spite of recognizing the visibility graph efficiently, [26] firstly introduced three necessary conditions for simple polygon visibility graphs, and conjectured that they are already sufficient. However, this conjecture is refuted by [23] via providing a counter-example and the third necessary condition was further strengthened in [24]. Unfortunately, [3] proved that these three conditions are still insufficient. Later, the fourth condition was proposed in [27], and together with the previous conditions, they are conjectured to be sufficient conditions as well. Again, it was disproved in [46]. Therefore, up to now, finding the necessary and sufficient condition for visibility graph are still continuing.

Chapter 3

Hidden Points and Hidden Vertices

A straightforward question arises here, what geometric element within a polygon helps in concealing more hidden points? Alternatively, how can we evaluate the potential capacity of a polygon to contain hidden points?

An initial answer would suggest that a polygon with more vertices would accommodate more hidden points. However, this is refuted by convex polygons, in which the number of vertices does not help at all. One might then consider manipulating the boundary by introducing reflex vertices or excavating holes, thereby creating obstacles to block the visibility. Intuitively speaking, both strategies seem promising, and we will try to formalize them in this chapter.

Before stepping into formal proofs, we must first establish the following necessary definitions.

3.1 Definition

Definition 3.1 *Let P be a simple polygon, $X \subseteq P$ be a point set in P.*

X is called a hidden point set in P if

$$\forall u \in X, v \in X, u \neq v, I_P(u,v) = 0.$$

Definition 3.2 *Let P be a simple polygon, $X \subseteq V(P)$ be a subset of vertices of P.*

X is called a hidden vertex set in P if

$$\forall u \in X, v \in X, u \neq v, I_P(u,v) = 0.$$

Definition 3.3 *Let P be a polygon, the hidden point number of P, denoted as $\mathrm{HP}(P)$, is size of maximum hidden point set of P.*

Definition 3.4 *Let P be a polygon, the hidden vertex number of P, denoted as $\mathrm{HV}(P)$, is size of maximum hidden vertex set of P.*

Indeed, $\mathrm{HV}(P)$ is the size of the maximum independent set of $VG(P)$, the visibility graph of P.

Proposition 3.5 *Let P be a simple polygon, we have $\mathrm{HV}(P) \leq \mathrm{HP}(P)$.*

Proof Since $V(P) \subseteq P$, any hidden vertex set is also a hidden point set.$\square$

Definition 3.6 *Let P be a simple polygon, a convex covering of P is a collection of finite number of convex polygons such that their union is exactly P. Denote any convex covering of P as $\mathcal{Q}$, then*

$$\begin{aligned} \mathcal{Q} = \ & \{Q_1, Q_2, \cdots, Q_k\}, \\ s.t. \ & \bigcup_{i=1}^{k} Q_i = P. \end{aligned}$$

Similarly, we can also define a convex partition of P by only allowing different Q_i share at most a common vertex or a common edge. Denote the convex partition of P as $\mathcal{Q}$, then

$$\begin{aligned} \mathcal{Q} = \ & \{Q_1, Q_2, \cdots, Q_k\}, \\ s.t. \ & \bigcup_{i=1}^{k} Q_i = P. \\ & Q_i \cap Q_j \subseteq \partial Q_i, \forall i \neq j. \end{aligned}$$

Accordingly, let $\textsc{Cover}(P), \textsc{Partition}(P)$ be the minimum size of the convex covering and convex partition of P respectively.

3.2 In Simple Polygon

This section is committed to illustrate some basic facts about the hidden points and the hidden vertices in simple polygons.

Lemma 3.7 *Let P be a simply polygon, then we have* $\mathrm{HP}(P) \leq \textsc{Cover}(P) \leq \textsc{Partition}(P)$.

Proof $\textsc{Cover}(P) \leq \textsc{Partition}(P)$ holds obviously as a convex partition is always a convex covering. It suffices to show that $\mathrm{HP}(P) \leq \textsc{Cover}(P)$.

Let $A \subseteq P$ be the maximum hidden point set. Let Q be the optimal convex covering of P and $Q = \{Q_1, Q_2, \cdots, Q_k\}$ and $|Q| = \textsc{Cover}(P) = k$.

Let $M_i = A \cap Q_i$. It is clear that $\forall i \in [k], |M_i| \leq 1$. Otherwise, suppose that there exist two distinct points $u, v \in M_i$. Since Q_i is convex, we have $\overline{uv} \subseteq Q_i \subseteq P$, implying that u and v are visible to each other.

Therefore, since $A = \bigcup_{i=1}^{k} M_i$, by union bound we have

$$\mathrm{HP}(P) = |A| \leq \sum_{i=1}^{k} |M_i| \leq k = \textsc{Cover}(P). \qquad \square$$

Definition 3.8 *Let $Q = \{Q_1, Q_2, \cdots, Q_k\}$ be a convex partition of P, the Steiner points of the convex partition Q are the extra vertices introduced by Q which are not vertices of the polygon P. In other words,*

$$Steiner_P(Q) = \bigcup_{i=1}^{k} V(Q_i) \setminus V(P).$$

Then, Q is said to be a convex partition without Steiner points if and only if $Steiner_P(Q) = \varnothing$.

Definition 3.9 *Let $Q = \{Q_1, Q_2, \cdots, Q_k\}$ be a convex partition of P, the planar graph induced by Q is defined as the planar graph represented by the drawing of Q. In other words, $G(Q) = (V, E)$, where $V := \bigcup_{i=1}^{k} V(Q_i)$ and $E := \bigcup_{i=1}^{k} E(Q_i)$.*

Lemma 3.10 *Let P be a simple polygon with n vertices,* $\mathrm{HP}(P) \leq n - 2$.

Proof By Lemma 3.12, it suffices to show that $\textsc{Partition}(P) \leq n - 2$.

Prove this by induction. For $n = 3$, it holds trivially. Suppose that it holds for $n = k$, and let us consider the case for $n = k + 1$.

By Two Ears Theorem, for any simple polygon $P = (p_0, p_1, \cdots, p_k)$, there exist $i \in [k-1]$, such that the triangle $(p_{i-1}, p_i, p_{i+1}) \subseteq P$. Let $Q = (p_0, p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_k)$, which is the polygon given by cutting the ear (p_{i-1}, p_i, p_{i+1}) from P. Notice that $P = (p_0, p_1, \cdots, p_k) = Q \cup (p_{i-1}, p_i, p_{i+1})$, and by induction, $\textsc{Partition}(Q) \leq k-2$. Thus we have $\textsc{Partition}(P) \leq \textsc{Partition}(Q) + 1 \leq k-1 = n-2$. $\square$

Lemma 3.11 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a simple polygon on n vertices, then* $\mathrm{HV}(P) \leq \lfloor n/2 \rfloor$.

Proof Prove this by contradiction. Suppose there exists a hidden vertex set $A \subseteq V(P)$ such that $2|A| > n$. Construct the corresponding B as follow,

$$B = \{p_{i+1} | p_i \in A\}.$$

Since $|A| + |B| = 2|A| > n \geq |A \cup B|$, we have $A \cap B \neq \varnothing$. Then, there exists $i \in [0, n-1]$ such that $p_i \in A$ and $p_{i+1} \in A$. Notice that p_i and p_{i+1} are visible to each other, they could not be selected in A simultaneously, leading to contradiction. $\square$

Lemma 3.12 *Let P be a simple polygon with r reflex vertices and n vertices in total, then* $\mathrm{HP}(P) \leq \textsc{Cover}(P) \leq \textsc{Partition}(P) \leq r+1$.

Proof By Lemma 3.12, it suffices to show that $\textsc{Partition}(P) \leq r+1$.

Let $Q = \{q_0, q_1, \cdots, q_{r-1}\} \subseteq V(P)$ be the reflex vertices of P. Start from q_0 to q_{r-1}, for each reflex vertex q_i, consider the ray $\overrightarrow{q_i t}$ which is the bisector of the interior angle of vertex q_i. Let w be the point on the ray $\overrightarrow{q_i t}$ where the ray intersect with other segments (including the segments introduced previously) at the first time.

There are two cases to consider.

- $\mathcal{A}$ w is a vertex that already exists, then we connect q_i to vertex w with segment $\overline{q_i w}$.
- $\mathcal{B}$ w is a interior point of the edge (u, v), then we introduce a new vertex w and split the edge into two edges (u, w) and (v, w), and still we connect q_i to the vertex w by $\overline{q_i w}$.

It is clear that such drawing is still a planar since no intersection is incurred. Then we can claim that the faces of the above straight-line planar drawing, except for the infinite face which includes the point at infinity, are already a convex partition of P.

Denote the planar graph as G, first we argue that each finite face of G is actually a convex polygon. Suppose the contradiction, there is a finite face f such that there exist a vertex $u \in V(f)$ and v is a reflex vertex of f.

If $v \in V(P)$, then there is an edge (u, v) such that (u, v) is the bisector of the interior angle of vertex v in P, which splits it into 2 convex interior angles, leading to a contraction.

If $v \notin V(P)$, this implies it is introduced during the above drawing. Then before we introduce it into G, it is a interior point of some edge e and is incident to at most two finite faces, where its interior angle is exactly π. Therefore, v can not be a reflex vertex in any face f, a contraction.

Since all the finite faces do not overlap and cover the polygon P, we conclude that it is a convex partition of P.

Let r_1 and r_2 denote the number of cases $\mathcal{A}$ and $\mathcal{B}$ respectively, and let V, E, and F denote the number of vertices, edges and faces in the planar drawing respectively. Thus, we have $V = n + r_2, E = n + r_1 + 2r_2$. By Euler's Formula, we have $V - E + F = 2$, thus $F = E - V + 2 = r_1 + r_2 + 2 = r + 2$. Notice that apart from exactly one infinite face, we have $r + 1$ finite convex faces, thus $\text{HP}(P) \leq \textsc{Partition}(P) \leq r + 1$. $\square$

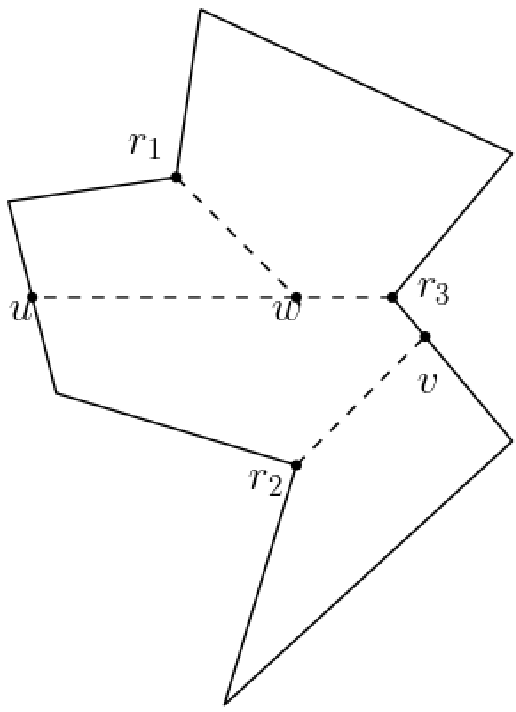


Figure 3.1: This figure illustrates the proof of Lemma 3.12, where the dashed lines are the introduced bisectors.

Apart from the size of maximum hidden point set, we are also interested in the locations of them. Consider such a scenario, you are asked to put as many hidden points as possible in the following polygon, and at the first glance, where would you put them at?

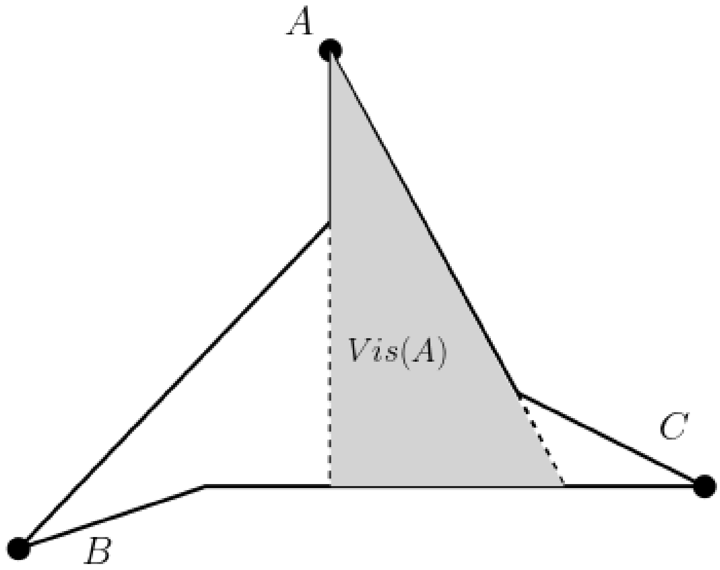


Figure 3.2: (A, B, C) is an optimal hidden points arrangement, the closed region filled by grey color is $Vis(A)$, the visible area from vertex A

It is likely that one may think that the convex vertex A is a good choice for hidden points because it is in a more isolated position. In other words, its visible areas $Vis(A)$, the region where points in it are visible from A, is "narrower" than the visible areas of points other than A. In the subsequent lemmas, we aim to formalize this intuition, providing sufficient conditions for points to be included in the maximum hidden point set.

Lemma 3.13 *Let P be a simple polygon and H be an maximum hidden point set. Let $p \in P$ be a point and $Vis(p) := \{q|\overline{pq} \subseteq P, q \in P\}$ be the visible area from point p. Then we have $H \cap Vis(p) \neq \varnothing$.*

Proof Prove this by contradiction and suppose that $H \cap Vis(p) = \varnothing$. Then, $H' = H \cup \{p\}$ is also a hidden point set as any point in H is invisible to the point p. Further, we have $|H'| > |H|$, leading to a contradiction against $|H| = \mathrm{HP}(P)$. $\square$

Lemma 3.14 *Let P be a simple polygon and $p \in P$ be a point such that $Vis(p)$ is convex. Then, there exists a maximum hidden point set H such that $p \in H$.*

Proof Let H' be a maximum hidden point set in P. By Lemma 3.13, there exists a point $q \in H'$, $q \in Vis(p)$. Notice that for each $r \in Vis(p)$, we have $\overline{qr} \subseteq P$ since $Vis(p)$ is convex. Thus, $Vis(p) \subseteq Vis(q)$ and $(H' \setminus \{q\}) \cap Vis(p) = \varnothing$. Therefore, we have $H = H' \setminus \{q\} \cup \{p\}$ is another maximum hidden point set which includes point p. □

3.3 In Polygon with Holes

In the last section, we have seen that $\mathrm{HP}(P) \leq r+1$, where r is the number of reflex vertices. Consequently, generally speaking, more reflex vertices help us host more hidden points in the polygon. Meanwhile, one may think that more holes in the polygon can hide more points, as they are natural "obstacles" in the polygon to block the lines of vision. In this section,we will explore the impact of the number of holes on the size of maximum hidden point set.

Lemma 3.15 *Let Q be a polygon with n vertices and h holes, then we have* $\mathrm{HP}(Q) \leq n-h-2$.

Proof Let $\mathcal{H} = \{\}$ Notice that each polygon has at least 3 convex vertices, then Q has at most $n-3$ reflex vertices. Analogous to Lemma 3.12, by introducing a bisector at each of them, we have a convex partition of the plane with $n-1$ convex polygons. Among them there are h holes, and an infinite face, which do not belong to Q. Therefore, $\mathrm{HP}(Q) \leq n-1-(h+1) = n-h-2$. □

From this we could see that, arranging vertices with the configurations with many holes, actually does not help with hiding more points, which may be a counter-intuition point. The secret here is that the vertices of the holes also are also counted in n. However, if we merely care about the number of holes, we can see that it does increase the capacity of hidden points in the following lemma.

Lemma 3.16 *Let P be a polygon with h rectilinear holes, then* $\mathrm{HP}(P) \geq \lfloor \sqrt{h} \rfloor$.

Proof For $1 \leq h \leq 4$, the cases are oblivious that $\mathrm{HP}(P) \geq 2$ because a simple polygon with at least a hole in its interior can not be convex.

For $h > 4$, let $m = \lfloor \sqrt{h} \rfloor$, and we will prove the lemma by giving an explicit arrangements of at least m hidden points.

We try to figure out the location of the hidden points by the following observation on rectilinear polygons.

Fact 3.17 *Let H_i be rectilinear polygon, then there exist three consecutive vertices $(a_i, b_i, c_i) \in H_i$ in counterclockwise order such that*

$$x_a = x_b < x_c,$$
$$y_a > y_b = y_c.$$

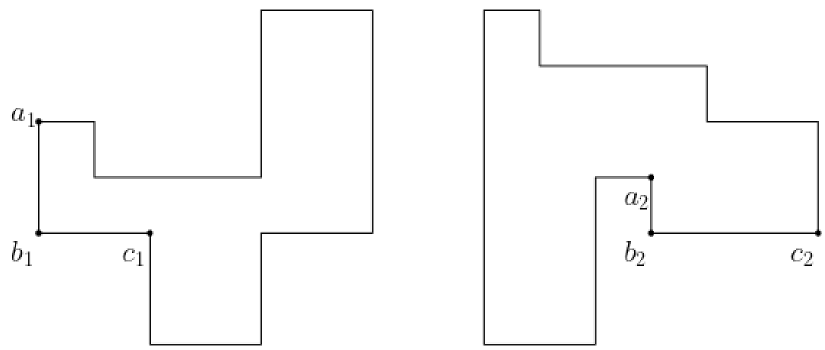


Figure 3.3: The marked vertices (a_i, b_i, c_i) are vertices of our interests in each polygon.

We call the vertex b_i the bottom-left corner of the rectilinear polygon H_i with $b_i = (x_i, y_i)$, and we further assume that $x_i \leq x_{i+1}$. We will find a subsequence of the bottom-left corners such that they are y-monotone by the following theorem.

Theorem 3.18 ***Erdős–Szekeres Theorem [22]**: Given a sequence b with at least $(k-1)(l-1)+1$ real numbers, there exists at least one of the followings:*

- *a subsequence of indices I with size k such that $b_{I_i} \leq b_{I_{i+1}}, \forall i \in [k-1]$,*
- *a subsequence of indices I with size l such that $b_{I_i} > b_{I_{i+1}}, \forall i \in [l-1]$.*

Let $k = l = m$, we have $(k-1)(l-1)+1 = (m-1)^2+1 \leq m^2 \leq h$. Therefore, by applying Theorem 3.18 on the sequence $\{b_i = (x_i, y_i) | i \in [h]\}$, we can see that we have a subsequence $I \subseteq [h]$ of length m such that $x_{I_i} \leq x_{I_{i+1}}$ and y_{I_i} is monotonely increasing or decreasing.

For any sequence λ, we rewrite λ_{I_i} as λ_i if we are discussing about the subsequence guaranteed by Theorem 3.18, to get rid of the double subscripts.

Suppose the subsequence y_i is monotonely increasing, in other words, $y_i \leq y_{i+1}$. Our goal is to place the hidden point u_i on each polygonal chain (a_i, b_i, c_i). Consider each $i \in [m-1]$, if $x_i < x_{i+1}$, then let u_i be the midpoint of (a_i, b_i), and we can see that $\forall j > i$, the point u_j on the the polygonal chain (a_j, b_j, c_j) is invisible to point u_i since $\overline{u_i u_j} \cap int(H_i) \neq \varnothing$.

Otherwise, if $x_i = x_{i+1}$, we have $y_i < y_{i+1}$ as the holes are disjoint and the corners can not overlap. Accordingly, we place the hidden point u_i on the midpoint of (b_i, c_i) and any point $u_j, j > i$ on (a_j, b_j, c_j) is invisible to u_i. For the last hidden point u_m, let $u_m = b_m$, thus completing the construction.

Otherwise, suppose the subsequence y_i is strictly monotonely decreasing, implying $y_i > y_i$. In this case, for each $i \in [m]$, we just let u_i be the midpoint of (b_i, c_i), composing a hidden point set.

Therefore, we can always place $m = \lfloor \sqrt{h} \rfloor$ hidden points, which means that $\mathrm{HP}(P) \geq \lfloor \sqrt{h} \rfloor$. □

Indeed, we will see that the lower bound $\sqrt{h}$ is asymptotically tight by the following proposition.

Proposition 3.19 *Let P be a polygon with h rectilinear holes, then there exist P such that* $\mathrm{HP}(P) \leq 2(\lceil \sqrt{h} \rceil + 1)$.

Proof We give the explicit arrangements of the rectilinear holes in P, illustrated by the following figure. Let $f(h) = \min_P \mathrm{HP}(P)$, where P is the polygon with h rectilinear holes. It is obvious that $f(h) \leq f(h+1)$. Thus, it suffices to prove the case where $h = m^2, m \in \mathbb{N}^+$ is a perfect square since for all $(m-1)^2 < h' < h$, we have $f(h') \leq f(h)$ and $\lceil \sqrt{h'} \rceil = \lceil \sqrt{h} \rceil$.

Let Q be a convex polygon and $m = \sqrt{h}$. Let all the rectilinear holes be squares of the same size and arrange them into a grid formation with m rows and m columns. Denote the square at the i-th row and the j-th column as $s_{i,j}$. For each $1 < i \leq m$, we make sure that the vertical edges of $s_{i,j}$ are collinear with the vertical edges of $s_{i-1,j}$. Similarly, for each $1 < j \leq m$, the horizontal edges of $s_{i,j}$ should be collinear with those in $s_{i,j-1}$ as well. Finally, let the desired polygon P be $Q \setminus \bigcup_{i,j} int(s_{i,j})$. See the following figure for detailed illustration.

It is clear that $\mathrm{HP}(P) \leq \textsc{Cover}(P) \leq 2(m+1)$, because P can be covered by $2(m+1)$ horizontal and vertical corridors, which lies between and beside the rows and columns, as is marked in the following figure 3.4.□

Actually, our investigation on this problem is not limited to this very special case. Instead, we try to step into different scenarios, by either restricting the type of holes or restricting the structure of polygons. In all the cases we already figured out, the lower bound of the hidden points is

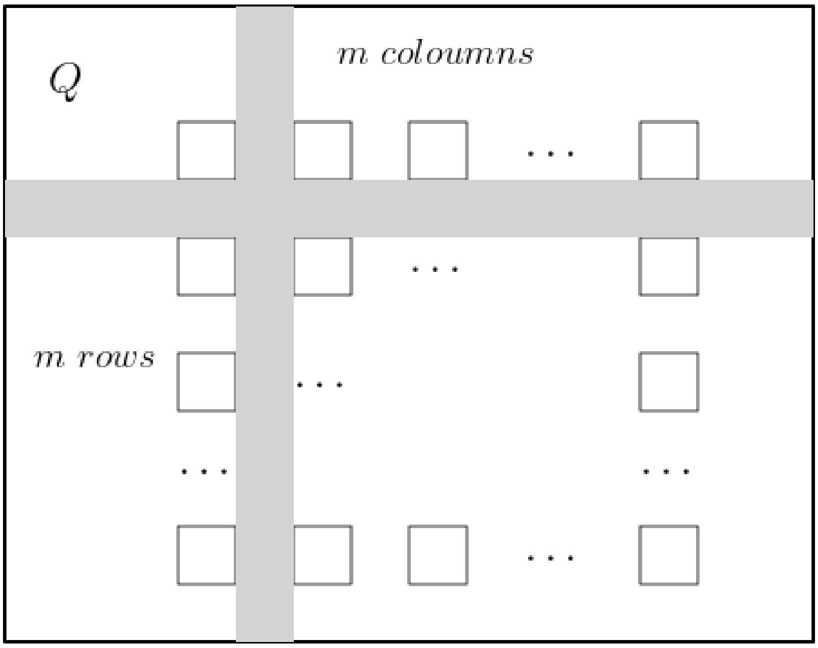


Figure 3.4: Polygon with $h = m^2$ holes. One horizontal corridor and another vertical corridor are filled with grey.

at least $\Omega(\sqrt{h})$. Therefore, it is reasonable to come up with the following conjecture.

Conjecture 3.20 *There exists constant $c > 0$, such that for any polygon P with h holes,* $\mathrm{HP}(P) \geq c\sqrt{h}$ *stands.*

Chapter 4

Set System of Visible Areas

Given a simple polygon P and a point $u \in P$, the task is to efficiently find another point v in P such that u and v are invisible, or report such a point v does not exist.

If this were an assignment in the Algorithm Lab, it is highly likely that a student would propose a solution involving the enumeration of vertices in P to find such a point v. This approach is indeed correct. However, what is intriguing is that while v itself can be located anywhere in P, we only consider the vertices as candidates for v. Thus, it becomes a question why are these candidate vertices already sufficient for finding v? Is there a scenario where all vertices of P are visible to u, but not every point in P? These questions is concerning the characterization of the visible area of u, which we will explore in this chapter.

Given a simple polygon P and a point u inside P, the visible area of u with regard to P is the set of points which are visible to point u, which is defined as the following.

Definition 4.1 *Let P be a simple polygon and point $u \in P$, the visible area from point u with regard to polygon P is defined as $Vis_P(u) := \{v | \ \overline{uv} \subseteq P\}$.*

In context of this thesis, if the polygon P is clear, we always omit the subscript of the visible area and write it as $Vis(u)$. Further, we always assume that $Vis(u)$ is regular simple polygon. That is its boundary is a Jordan curve. This assumption is only for the simplicity of writing and reading. The propositions we would argue about indeed stands for the cases where the visible area is an irregular polygon, but the proofs of them may involve additional notations.

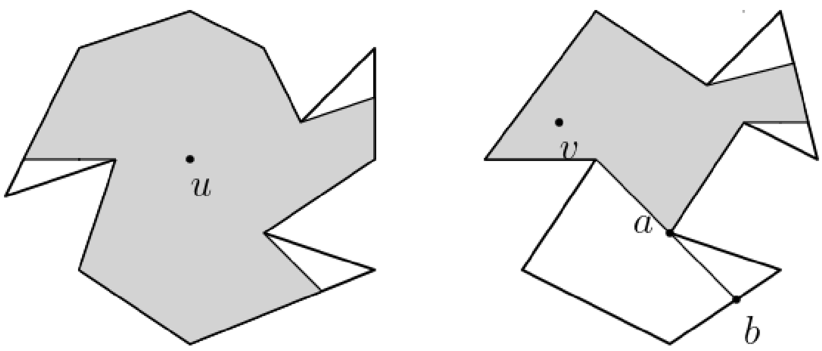


Figure 4.1: In these figures, the grey areas represent the visible area of point u and point v respectively. Notice that, $Vis(u)$ is a regular simple polygon, but $Vis(v)$ is not since the segment $\overline{ab}$ also belongs to it.

In the last chapter, especially in Lemma 3.13 and Lemma 3.14, we could see that the arrangement of the hidden points is closely related to the property of the visible areas, and that is why we are interested in them. In addition the connection between them, we are also willing to describe the structure of the visible area itself, which this chapter is committed to.

By Definition 4.1, we could define the set system of the visible areas.

Definition 4.2 *Let P be a simple polygon, the set system of its visible areas, denoted as $VS(P)$, is defined as $VS(P) := (P, S)$, where $S := \{Vis(u) | u \in P\}$.*

Actually, in the language of visible areas, the hidden point problem is to locate these points in the polygon such that each of their visible areas excludes any other point.

4.1 Steiner Points and Windows in Visible Areas

In this section, we are ready to propose basics facts about the visible area, in particular, what are its vertices and edges composed of.

Proposition 4.3 *Let P be a simple polygon, u be a point in P and $Q = Vis(u)$, then $V(Q) \subseteq \partial P$. That is, all the vertices of Q lie on the boundary of P.*

Proof Prove this by contradiction. Suppose the opposite, there exists a vertex v of Q such that $v \in int(P)$.

First we claim that $u \neq v$. Otherwise, suppose $u = v$, then there exists ball $B(u, \varepsilon), \varepsilon > 0$ such that $B(u, \varepsilon) \subseteq int(P)$. Hence, we have $B(u, \varepsilon) \subseteq Q$. However, notice that $u = v \in \partial Q$, so for any $\varepsilon > 0$, we have $B(u, \varepsilon) \nsubseteq Q$, thus leading to the contradiction.

Then, let's consider the two edges in Q which are incident to the vertex v. At least one of them is not collinear with $\overline{uv}$, otherwise v is no longer a vertex of Q. Suppose that (v, w) is the edge of Q and points u, v, w are not collinear. Let r be a relative interior point of $\overline{vw}$. By the assumption $v \in int(P)$, we have $r \in int(P)$.

Further, let s be the point where the ray $\overrightarrow{ur}$ intersect with ∂P or ∂Q at the first time, apart from the point r. Let t be a relative interior point of $\overline{rs}$, then we have $t \notin Q, t \in int(P)$. However, because $\overline{ut} \subseteq P$ and point t is visible to point u, t must be in Q, leading to the contradiction.

Therefore, for any vertex v of Q, $v \in \partial P$. $\square$

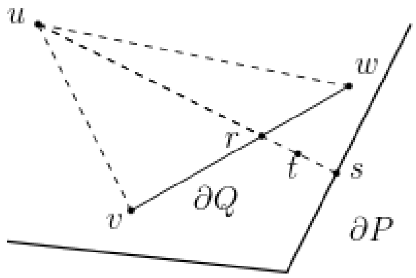


Figure 4.2: This figure illustrates the proof of Proposition 4.3.

Let P be a simple polygon and Q be a fixed visible area in P. Proposition 4.3 shows that all the vertices of Q lie on the boundary of P. However, not all of them are vertices of P, and some of them may lie in the relative interior of edges in P. Also, some edges of Q are not arranged through the boundary of P, as they might cross the interior of P.

Definition 4.4 *Let P be a simple polygon, u be a point in P and $Q = Vis(u)$, a vertex v of Q is called a Steiner point if $v \notin V(P)$, and an edge e of Q is called a window if $e \nsubseteq \partial P$.*

Proposition 4.5 *Let P be a simple polygon and Q be the visible area of a point u in P. Let $e = (v, w)$ be an edge of Q, then at least one of the following stands:*

1. *$e \subseteq \partial P$.*
2. *points u, v, w are collinear.*

Proof Suppose the opposite, there exists an edge $e = (v, w)$ of Q such that $e \nsubseteq \partial P$ and points u, v, w are not collinear.

Let r be the midpoint of edge e, since $e \nsubseteq \partial P$, we have $r \in int(P)$. Further, $\overrightarrow{ur}$ are not collinear with $\overrightarrow{uv}$ and $\overrightarrow{uw}$. Let s be the point where

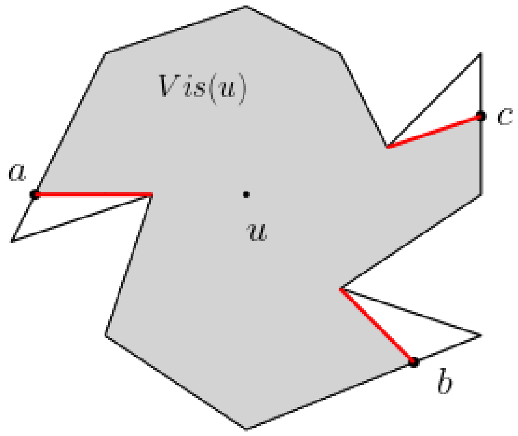


Figure 4.3: As is showed in this figure, point a, b and c are Steiner points introduced by $Vis(u)$ and the red segments are the windows of $Vis(u)$.

$\overrightarrow{ur}$ intersect with ∂P or ∂Q at the first time, apart from the point r, let t be the midpoint of the segment $\overline{rs}$. Thus we have $t \in P$ and $t \notin Q$. However, since $\overline{ut} \subseteq P$, t should be visible to point u, which gives rise to the contradiction. $\square$

As is directly implied from the proposition 4.5, for each window of the visible area, it must be collinear to the central point u.

Corollary 4.6 *Let P be a simple polygon and $Q = Vis(u)$. Let $e = (v, w)$ be a window of Q, then e is incident to at most one Steiner point.*

Proof Suppose that e is incident to two Steiner points. Suppose that $v \in int(z_1), w \in int(z_2)$, where z_1 and z_2 are the edges of P.

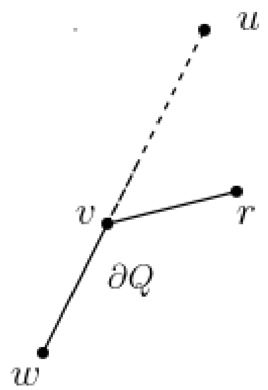
Let l be the line passing e, by Proposition 4.5, we have $u \in l$. Notice that l properly intersects with z_1 and z_2, so $u \notin l \subseteq e$ and $u \in e$.

If $u = v$, z_1 is visible to u, and v is no longer a Steiner point, a contradiction. Therefore, we have $u \neq v$ and $u \neq w$.

If $u \in int(e)$, u is indeed a interior point P, and there exists $B(u, \varepsilon) \subseteq P, \varepsilon > 0$, thus making $B(u, \varepsilon) \subseteq Q$. However, since $u \in e \subseteq \partial Q$, we should have $B(u, \varepsilon) \nsubseteq Q$, leading to a contradiction. $\square$

Proposition 4.7 *Let P be a simple polygon and Q be the visible area of the point u in P. Let $e = (v, w)$ be the window of Q and v be the endpoint that is closer to u. Thus we claim that v is a reflex vertex in both P and Q.*

Proof We already know that u, v, w are collinear. By the arguments in Corollary 4.6, we know that $u \notin \overline{vw}$. We assume v is the endpoint which is closer to point u, as is showed in the following figure.



Let r be the vertex succeeding the vertex w and v in Q. Notice that r can not be collinear with e since v is a vertex in Q. Then we can see that v is indeed a reflex vertex since $\angle wvr = \angle wvu + \angle uvr = \pi + \angle uvr > \pi$.

Hence, since v is a reflex vertex of Q, it must also be a reflex vertex of P, as all the Steiner points are convex vertices. $\square$

Accordingly, each window is incident to a reflex vertex in the original polygon. Additionally, each Steiner point is also incident to a window, otherwise it can not become a vertex of the visible area. Thus, we can conclude the upper bound on the number of them.

Lemma 4.8 *Let P be a simple polygon with r reflex vertices and Q be the visible area of point u in P. Then Q has at most r Steiner points and at most r windows.*

Proof It suffices to show that Q has at most r windows, as each Steiner point must be incident to a window.

Let $e = (v, w)$ be a window in Q and suppose v is the endpoint closer to point u. By Proposition 4.7, we can see that v is a reflex vertex in P, and we say that window u is associated to this vertex v (the closer vertex).

Further, we can see that each window is associated to a unique reflex vertex. Otherwise, suppose that two windows e_1 and e_2 are associated to the same reflex vertex u. However, by Proposition 4.5, u is collinear to both e_1 and e_2. This is actually impossible because e_1 and e_2 can not be collinear, otherwise u is no longer a vertex of the visible area.

Therefore, there are at most r windows in Q, and thus at most r Steiner points. $\square$

Lemma 4.9 *Let P be a simple polygon with n vertices and Q be the visible area of point u in P. Then Q has at most n vertices.*

Proof Suppose there are s Steiner points in Q, and let $X = V(P) \cup V(Q)$, then $|X| = n + s$. Let $X = (p_0, p_1, \cdots, p_{n+s-1})$ be the cycle composed of these vertices in the counter-clockwise order.

Suppose $p_0 \in V(Q)$ and consider the following scenario. We start with p_0 and visit all the vertices of Q among X in counter-clockwise order, and eventually back to p_0, giving us a subset of X. Let $e = (v, w)$ be a window of Q, and thus they are not adjacent in X. Given that we need to visit w immediately after v, we must bypass some vertices between them in X. When walking through these s windows, we have to make s "leaps", which excludes at least s vertices.

Therefore, $|V(P) \setminus V(Q)| \geq r$, $|V(Q)| \leq n + s - s = n$.

Lemma 4.10 *Let P be a simple polygon on n vertices, and u be a point in P. Then, $Vis_P(u)$ can be computed in $O(n)$ time.*

Proof This is proved jointly by [31] and [13]. □

In summary, this section discusses about the basic facts about the single visible area, which paves our way to the set system of the visible areas.

Remark 4.11 *All the above statements stands even if the visible areas are not regular simple polygons, but in our proofs we assume they are, merely for brevity.*

4.2 VC-dimension of Visible Area Set System

In terms of a set system, what naturally draws our attention is the Vapnik–Chervonenkis (VC) dimension, which evaluates the complexity and flexibility of a set system. In this section, given a simple polygon P with r reflex vertices, we prove that its set system of visible areas has VC-dimension at most $O(\log r)$.

Definition 4.12 *Let U be a set and S be a family of subsets of U. Let X be a subset of U. X is said to be shattered by S if and only if for any $Y \subseteq X$, there exist $C \in S$ such that $C \cap X = Y$.*

In our context, U is also called the ground set and the pair (U, S) is the set system.

Definition 4.13 *Let X be the ground set and $\Lambda = (X, S)$ be a set system over X. The VC-dimension of Λ, denoted as $dim(\Lambda)$, is defined by:*

$$dim(\Lambda) = \begin{cases} -1 & \textit{if } S = \varnothing, \\ k & \textit{if } k \textit{ is the size of the largest sets in } F, \\ +\infty & \textit{if there exists arbitrary large set in } F, \end{cases}$$

where F is the collection of the subsets of X which can be shattered by S.

Let P be the simple polygon, which is the ground set, and $\Lambda = VS(P)$ be the set system of visible areas in P. The VC-dimension of Λ is the maximum number of points that we can arrange in P such that for any subset of them, we can locate an observer in P which is visible to this subset but invisible to its complement. In the following paragraphs, we will show that actually we can not arrange too many such points.

Lemma 4.14 *Let P be a simple polygon with r reflex vertices and $\Lambda = VS(P) = (P, S)$, then we have $dim(\Lambda) \leq 10 \log r$.*

Proof If $dim(\Lambda) = 0$, the lemma stands certainly. If $r = 0$, we have P is convex and $S = \{P\}$. Accordingly, $dim(\Lambda) = 0$ and the case is also trivial. Thus, we assume that both $dim(\Lambda)$ and r are positive.

Let $X = \{u_1, u_1, \cdots, u_k\} \subseteq P$ be the set of k points can be shattered by S. Further, for each point u_i, let $Q_i = Vis(u_i)$ denote its visible area. Draw the polygon P and each $Q_i, i \in [k]$ on the plane, and denote this plane graph as G. The vertices of G are composed of two parts: One is from the original polygon P and visible areas Q_i, and another is the points where edges of the visible areas properly intersect. Accordingly, the edges of G are subdivisions of the original edges in P and Q_i.

Let v, e and f denote the number of vertices, edges and faces of this planar drawing. Notice that X can be shattered by S. Let Y be each subset of X, and we can find a point in P such that it is visible to Y and invisible to $X \setminus Y$, thus giving us in total 2^k points. Further, any two of them must reside in different faces, as the point in the same face are also visible to the same part of X. Take the infinite face into account, we should have the lower bound of f, $f \geq 2^k + 1$.

By the Euler's formula $v - e + f = 2$, we have $f = 2 + e - v$. Further, let s denote the number of Steiner points in all the visible area of u_i. Thus, we have $v \geq n + s$, which is bounded by number of vertices on ∂P.

Meanwhile, let's derive the upper bound of e. The edges of G are composed of two parts: One lies on ∂P, which has in total $n + s$ edges, and another cross the interior of P, which originates from the windows of Q_i. Notice that for each visible area Q_i, it has at most r windows, thus we have at most kr windows in total. For each window in Q_i, it can only properly intersect with the windows from the other visible area $Q_j, j \neq i$. Therefore, each window can contribute $(k-1)r + 1 \leq kr$ edges. Aggregate them together, we have $e \leq n + s + k^2r^2$.

Then, we have the upper bound of f, which is $f \leq 2 + k^2r^2$. Combine it with the lower bound, we have $2^k + 1 \leq 2 + k^2r^2$, implying that $k \leq 10 \log r$.

Therefore, we can conclude that Λ has VC-dimension at most $10 \log r$.□

It is not satisfying that the set system of visible areas has VC-dimension bounded by the logarithm of number of reflex vertices. Further, it is also conjectured to be upper bounded by some constant, which is formulated in the next conjecture.

Conjecture 4.15 *There exists constant $C > 0$, such that for any simple polygon P, let Λ be its set system of visible areas, we have $dim(\Lambda) < C$.*

Chapter 5

Hardness of Hidden Points

Linear programming is one of the most famous mathematical optimization problems. Despite dealing with continuous real space, it is well-known that an optimal solution can always be found among the vertices of the polytope, which can be formulated as a discrete combinatorial structure. Further, it can be showed that it always has a polynomial size optimal solution, thus making itself contained in NP. However, not all mathematical optimization problems enjoy the same fortune. The celebrated art gallery problem, for instance, has been proved that sometimes irrational coordinates are necessary [4].

When it comes to the maximum hidden points problem, a similar question persists, and we will step into it further in this chapter.

5.1 NP Optimization Compendium

When we talk about the hardness of a problem, we usually refer to the hardness of a decision problem. Given a language L and an instance x, we care about if we can decide $x \in L$ or $x \notin L$ in polynomial time with a deterministic/non-deterministic Turing machine. However, in the following contexts, we will use a completely different complexity compendium. That is the hardness of the optimization problems, which was originally proposed in [16].

Definition 5.1 ***NP Optimization problem***: *An NP optimization problem A (in abbreviation* **NPO** *problem) is defined as a fourtuple $(I, sol, m, goal)$, in which*

- *I is the set of instances of A and it is recognizable in polynomial time*
- *Let x be an instance of I, $sol(x)$ is the set of feasible solution of x, and there exists a polynomial p such that $\forall x \in I, y \in sol(x)$, we have $|y| \leq p(|x|)$. Further, $\forall x, y$ with $|y| \leq p(|x|)$, whether $y \in sol(x)$ or not can be decided in polynomial time.*
- *Let $x \in I$ and $y \in sol(x)$, $m(x,y)$ denotes the positive integer which is the measure evaluating the quality the solution. Further, $\forall x \in I, y \in sol(x)$, $m(x,y)$ can be computed in polynomial time.*
- *$goal \in \{max, min\}$, which describe our primary objective is to either maximize or minimize the measure m.*

The goal of A with respect to the given instance $x \in I$ is to find an optimal solution y^* such that

$$m(x,y^*) = goal\{m(x,y)|y \in sol(x)\}.$$

Accordingly, the optimal value $opt(x)$ is defined as $opt(x) := m(x,y^*)$.

Generally speaking, the main difference between NP Optimization problem and general mathematical optimization is that NP Optimization only recognize polynomial large solutions. Given an instance x and a solution y, we can decide whether x is a valid instance, whether y is a feasible solution, and further evaluate the performance of y in polynomial time.

Let us make the maximum hidden vertex problem as an example.

Let $A = (I, sol, m, gol)$ be a NPO problem for the hidden vertex set problem, where

- I is the set of simple polygons. For any instance $x \in I$, x is represented by the coordinates of the n vertices, $x = (p_0, p_2, \cdots, p_{n-1})$. We can see that we can verify whether they compose a simple polygon or not in $O(n^2m^2)$, where m is the largest number of bits p_i has. Notice that $|x| = \Omega(\max\{n, w\})$, and thus I is recognizable in polynomial time.
- sol is the collection of the hidden vertex sets. Given the instance x with n vertices, a solution y is 0/1-bit string of length n such that for all $0 \leq i < n$, the vertex p_i is included in the corresponding set Y if and only if $y_i = 1$. Hence, we can verify $y \in sol(x)$ by checking whether Y is indeed a hidden vertex set

in polynomial time by computing the visibility graph of x.

- Given the instance x and a feasible solution y, $m(x, y)$ is the metric which counts the number of 1s in y, which is essentially the size of Y. Therefore, m can be computed in linear time $O(n)$ and hence upper bounded by n.
- $goal = max$ since our primary objective is to maximize the size of hidden vertex set.

Therefore, we can safely conclude that the maximum hidden vertex problem is an NP optimization problem.

5.2 Optimal Solution of Hidden Points

5.2.1 Irrational Points are not Necessary.

When it comes to the hidden points, the case becomes much more difficult than hidden vertices. The key point is that, the feasible arrangement of hidden points in a given polygon (i.e. the coordinates of the hidden points), is not guaranteed to have polynomial size, given that the numeral system to represent the coordinates is fixed.

In this section, we can see that there is always an optimal solution for the hidden point set, whose coordinates are all rational.

At the very beginning, we need to address the first concern: is it possible that there exist an instance in which the optimal solution for maximum hidden point set is unique? In fact, we prove that such optimal solution is always not unique.

Lemma 5.2 *Let P be a simple polygon, then the maximum hidden point set in P is not unique.*

Proof Let P be a simple polygon and $A = (a_1, a_2, \cdots, a_k)$ be the maximum hidden point set in P. Let $Q_i = Vis(a_i)$, which is a closed set, and thus $\overline{Q_i}$ is a open set. Given fixed $\{a_2, a_3, \cdots, a_k\}$, for each $i \in [2, k]$, since $a_1 \in \overline{Q_i}$, there exists $\varepsilon_i > 0$, such that $B(a_i, \varepsilon_i) \subseteq \overline{Q_i}$.

Let $\sigma = \min\{\varepsilon_i\}_{i=2}^{k}$, then $\forall i \in [2, k], B(a_1, \sigma) \cap Q_i = \varnothing$. Since $\{a_1\} \subsetneq B(a_1, \sigma) \cap P$, there exists $a_1' \neq a_1$, such that $a_1' \in B(u, \sigma) \cap P$. Therefore, $\{a_1', a_2, \cdots, a_k\}$ is another maximum hidden point set other than A, indicating that the maximum hidden point set is not unique. $\square$

Take one step further, we can show that there always exists a rational optimal solution.

Proposition 5.3 *Let P be a simple polygon and Q be a simple polygon such that $V(Q) \subseteq \partial P$ and $Q \subseteq P, P \setminus Q \neq \varnothing$, then we have $int(P \setminus Q) \neq \varnothing$.*

Proof Notice that $V(Q) \subseteq \partial P$ and $Q \subsetneq P$, then Q has at least an edge e such that $e \subseteq \partial P$. Otherwise, we have $Q = P$, contradicting to $P \setminus Q \neq \varnothing$. Let $e = (u, v)$ be such an edge of Q and $P(u, v)$ be the polygonal chain alongside ∂P connecting u and v, which does not includes any other vertices from Q. Then we can see that e and $P(u, v)$ constitute a Jordan curve, encircling a simple polygon R. Hence, $int(R) \neq \varnothing$ and $int(R) \subseteq int(P \setminus Q)$. Therefore, $int(P \setminus Q) \neq \varnothing$. □

Corollary 5.4 *Let P be a simple polygon and u be a point inside P such that $P \neq Vis(u)$, then $int(P \setminus Vis(u)) \neq \varnothing$.*

Proof This is implied by $V(Vis(u)) \subseteq \partial P$ in Proposition 4.3 and Proposition 5.3. □

Lemma 5.5 *Let P be a simple polygon, then there exists an maximum hidden point set $B = (b_1, b_2, \cdots, b_k)$ such that $\forall i \in [k]$, $b_i = (x_i, y_i)$ and both x_i and y_i are rational numbers.*

Proof Let $A = (a_1, a_2, \cdots, a_k)$ be a maximum hidden point set in P, we will find a hidden point set such that it has the same cardinality as A and all of its coordinates are rational.

If $\forall i \in [k]$, $a_i = (x_i, y_i)$ is composed of rational coordinate, we already have it. Otherwise, suppose that $a_1 = (x_1, y_1)$ include irrational coordinates. Consider the feasible region of the hidden point a_1, $a_1 \in P \setminus Q$, where $Q = \bigcup_{i=2}^{k} Vis(a_i)$. By Corollary 5.4, we have $int(P \setminus Q) \neq \varnothing$.

Let the point $u \in int(P \setminus Q)$, and then there exists constant $\varepsilon > 0$ such that the l_1-ball $B_1(u, \varepsilon) \subseteq P \setminus Q$. Let m_1 be the smallest integer such that $\varepsilon \geq \frac{1}{m_1}$. Notice that $B_1(u, \varepsilon)$ is a axis-aligned square with length $2\varepsilon \geq \frac{2}{m_1}$, then there exist a point rational point $v = (X_1/m_1, Y_1/m_1)$ and $v \in B_1(u, \varepsilon)$ such that X and Y are integers. Consequently, $A' = v, a_2, \cdots, a_k$ forms another maximum hidden point set, with v being a rational point.

Thus, for each $i \in [k]$, by fixing the other hidden points, we can relocate hidden point a_i to a rational feasible point, thereby constructing the desired solution. □

In summary, the preceding paragraphs establish the feasibility of representing the hidden point set using a finite number of bits, which we confirm to be achievable.

5.2.2 Is the Hidden Point Problem in NPO?

This subsection is dedicated to discussing whether the hidden point problem is in the complexity class of NPO, which means an optimal solution of polynomial size can always be found. As of now, this conjecture remains undetermined, but we may present arguments that might suggest its validity.

Before we start, let us formally define the "size" of the number, which we already informally used in last subsection.

Definition 5.6 *Let n be an integer, the size of n is defined as the least positive integer l such that $-2^{l-1} \leq n \leq 2^{l-1} - 1$, denoted as $size(n)$.*

Definition 5.7 *Let $r = p/q$ be a rational number, where p is an integer, q is a positive integer, and p, q are coprime (such pair of p and q is unique). Then, the size of r is defined by $size(r) := size(p) + size(q)$.*

If the rational number r is of size l, we say that r has l bits. Besides, any irrational number is considered to have infinite large bits. Accordingly, we could further refine the arguments in last subsection.

Proposition 5.8 *Let x and y be two rational numbers with n and m bits respectively, and then $x + y$, $x - y$, $x * y$ and x/y have at most $2(n + m)$ bits.*

Definition 5.9 *Let $A = (x, y)$ be a point in the plane, A is called an integral point if both x and y are integers. Let P be a simple polygon, P is called an integral polygon if all of its vertices are integral points.*

Lemma 5.10 *Let P be an integral polygon and the coordinates of its vertices have at most m bits, then there exists a point $r \in int(P)$ such that coordinates of r have at most $m + 5$ bits.*

Proof Let Q be the smallest (in terms of area) integral triangle inside P. Notice that P must contain some integral triangle as the triangulation of P is composed by integral triangles. Since P only includes finite number of integral triangles, such Q always exists. It suffices to show that we can find a point $r \in int(Q)$ such that r has at most $m + 5$ points.

Further, we know that Q does not include any integral points other than its vertices. Otherwise, we could find a integral triangle inside Q which is strictly smaller.

By Pick's theorem [38], the area A of an integral polygon P is: $A = i + b/2 - 1$, where i is the number of integral points in $int(A)$ and b is the number of integral points in ∂P. Consider the triangle Q, we have $i_Q = 0$, $b_Q = 3$ and $A_Q = \frac{1}{2}$.

Hence, consider the following affine transformation in the plane:

$$f : (x, y) \to (3x, 3y),$$

which actually scales up the coordinates by three times. Suppose R is the image of Q under the mapping f. It is clear that R is also an integral triangle and $A_R = 9A_Q = \frac{9}{2}$.

Further, for each edge of R, it has only two integral points in its relative interior, which are the trisection points of it. Thus, $b_R = 9$. Since $A_R = i_R + b_R/2 - 1$, we have $i_R = 1$. This implies that there exists a integral point $u \in int(R)$. Let $r = f^{-1}(u)$, and we have $r \in int(Q)$.

Let $u = (x_u, y_u)$ and $r = (x_r, y_r)$. Since the coordinates of the vertices of Q have at most m bits, we have $|x_r| \leq 2^{m-1}$, thus $|x_u| = 3|x_r| < 2^{m+1}$, then $size(x_u) \leq m + 2$ bits. Hence, $x_r = \frac{x_u}{3}$, $size(x_r) = size(x_u) + size(3) \leq m + 2 + 3 = m + 5$. Therefore, x_r has at most $m + 5$ bits, and so does y_r.

In summary, we can find a point r in the interior of P such that the coordinates of r has at most $m + 5$ bits. $\square$

Corollary 5.11 *Let P be a simple polygon such that the coordinates of its vertices have at most m bits (do not necessarily to be integers). Then we can find a point $u \in int(R)$ such that the coordinates of r have $O(m)$ bits.*

Proof We suppose that P is a triangle. If not, let $\mathcal{T}$ be a triangulation of P and continue our argument on any triangle in $\mathcal{T}$.

Let $P = (p_0, p_1, p_2)$, and $p_i = (a_i/b_i, c_i/d_i)$.

Consider the following affine transformation in the plane:

$$f : (x, y) \to (b_0 b_1 b_2 x, d_0 d_1 d_2 y).$$

Let Q be the image of P under the mapping f. It is clear that Q is indeed an integral triangle and the coordinates of its vertices have $O(m)$ bits. By

Lemma 5.10, there exists a interior point $u \in Q$ such that coordinates of u have $O(m)$ bits.

Let $u = (x_u, y_u)$ and $r = f^{-1}(r)$. Thus we have $r = (\frac{x_u}{b_0 b_1 b_2}, \frac{y_u}{d_0 d_1 d_2})$ and $r \in int(P)$. Since x_u and y_u both have $O(m)$ bits, we can conclude that coordinates of r also have $O(m)$ bits. □

Lemma 5.12 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a simple polygon and $A = (a_1, a_2. \cdots, a_k)$ be an arrangement of hidden points such that:*

- *the coordinates of vertices in P and hidden points in A have at most m bits.*
- *$P \setminus (\bigcup_{i=1}^{k} Vis(a_i)) \neq \varnothing$.*

Then, we can find another point $a_{k+1} \in P$ such that $A \cup \{a_{k+1}\}$ is still hidden point set, and the coordinates of a_{k+1} have at most $O(m)$ bits.

Proof Let Q_i denote $Vis(a_i)$ and $R = P \setminus (\bigcup_{i=1}^{k} Q_i)$. Note that R has finite number of connected components, since by Lemma 4.9, each visible area has at most n edges, and thus leading to finite number of faces. Let C be the closure of any connected component of R.

Let's consider the boundary of C. For any edge e of C, exact one of the following statement stands:

- e is part of the boundary of P,
- e is part of the window of some Q_i.

Let l be the straight line passing through the edge e. If $e \subseteq \partial P$, l must pass through two consecutive vertices p_j and p_{j+1} in P. If e is part of the window of the visible area Q_i, by Proposition 4.5, l must pass by the hidden point a_i and a reflex vertex u of P. Therefore, let $X = V(P) \bigcup A$, we can conclude that l must pass through two distinct points in X.

Further, consider any vertex $w = (x_w, y_w)$ of the polygon C, it should be the intersection point of lines l_1 and l_2. By arguments before, we know that both l_1 and l_2 pass through at least two distinct points in X. Let $u = (x_u, y_u)$ and $v = (x_v, y_v)$ be the two points l_1 passing by in X. Then, the equation of l_1 can be formulated by:

$$l_1 : (x - x_u)(y_v - y_u) - (y - y_u)(x_v - x_u) = 0.$$

Similarly, let $s = (x_s, y_s)$ and $t = (x_t, y_t)$ the points l_2 passing by in X. Thus, the coordinates of w is the solution of the following linear system:

$$l_1 : (x - x_u)(y_v - y_u) - (y - y_u)(x_v - x_u) = 0,$$
$$l_2 : (x - x_s)(y_t - y_s) - (y - y_s)(x_t - x_s) = 0.$$

Since the coordinates of point u, v, s, and t have at most m bits, by Proposition 5.8, both x_w and y_w have $O(m)$ bits. Therefore, the coordinates of the vertices of C have at most $O(m)$ bits. By Corollary 5.11, we can find a point $r \in int(C)$ such that r also has $O(m)$ bits.

Notice that $r \in int(C)$ and $int(C) \subseteq R$, we have $r \in P \setminus (\bigcup_{i=1}^{k} Vis(a_i))$. Therefore, $a_{k+1} = r$ can be the additional hidden point, the coordinates of which only have $O(m)$ bits. $\square$

The above proposition shows that given the chance to place another hidden point, it is possible to locate it at some coordinate which only need number of bits that is "proportional" to other vertices and hidden points. But, this is not sufficient to guarantee us that the hidden point problem is in NPO, as such argument might lead to exponentially large solution in total.

5.3 Approximation for NP Optimization

Apart from the hardness of exactly solving the NP optimization problem, we are also interested in find a solution with relatively good performance, that is where the definitions of performance ratio and approximation algorithm come in.

Definition 5.13 *Let A be an* NPO *problem. Given an instance $x \in I$ and $y \in sol(x)$, $R(x, y)$ denotes the performance ratio of y with regard to x, which is defined as:*

$$R(x, y) = \max\{\frac{m(x, y)}{opt(x)}, \frac{opt(x)}{m(x, y)}\}.$$

Definition 5.14 *Let $A = (I, sol, m, goal)$ be an* NPO *problem, and T be an algorithm, then T is an approximation algorithm of problem A if $\forall x \in I$, we have $T(x) \in sol(x)$.*

Further, we can define the approximability of an *NPO* problem.

Definition 5.15 *Let A be an* NPO *optimization problem, and r be a function, and $r : \mathbb{N} \to (1, +\infty)$. Then, A is called $r(n)$-approximable if and only if there* *exists a* ***polynomial*** *approximation algorithm T of problem A such that*

$$\forall x \in I, R(x, T(x)) \leq r(|x|).$$

Accordingly, such T is called an $r(n)$-approximation algorithm for problem A.

According to the approximability, we can define the following subclasses of NPO.

Definition 5.16 *Let A be an* NPO *problem, then A belongs to the class* APX *if there exists some constant $c > 1$ such that A is c-approximable.*

Definition 5.17 *Let A be an* NPO *problem, then A belongs to the class n^{ε}-APX if there exists some constant $\varepsilon > 0$ such that A is n^{ε}-approximable.*

Definition 5.18 *Let A be an* NPO *problem, then A belongs to the class $log(n)$-APX if there exists some constant $c > 0$ such that A is $c \log(n)$-approximable.*

We can see that as the asymptotic bound of $r(n)$ increases, the scope of the class $r(n)$-APX also becomes larger, and it is more likely to contain harder NP optimization problems. The following figure 5.1 is an illustration of this hardness hierarchy. We remark that P in the diagram denote the set of NPO problems that can be solved in polynomial time.

Notice that this leveled structure stands given that we assume P $\neq$ NP. Otherwise, P = NPO is immediately implied[1] and this hardness hierarchy will collapse accordingly.

Among all the NP optimization problems, we are ready to define the completeness via the idea of the PTAS reduction. In one sentence, PTAS-reduction is a kind of reduction such that the performance ratio is preserved through the bijection of an invertible function h.

Definition 5.19 [16] *Let $A = (I_A, sol_A, m_A, goal_A)$ and $B = (I_B, sol_B, m_B, goal_B)$ be two* NPO *problems, then A is* PTAS *reducible to B if and only if there exists three function f, g, h such that:*

- $f : I_A \times \mathbb{Q} \cap (1, +\infty) \rightarrow I_B$.
- *For any $x \in I_A$ and any rational $\varepsilon \in (1, +\infty)$, we have $f(x, \varepsilon) I_B$, and it can be computed in polynomial time with regard to $|x|$.*

[1]To see this, we can do binary search on the output of objective function m. The corresponding decision problem is in NP, and by assumption can be solved in polynomial time. Note that m is computable in polynomial time, so it take at most polynomial times binary search.

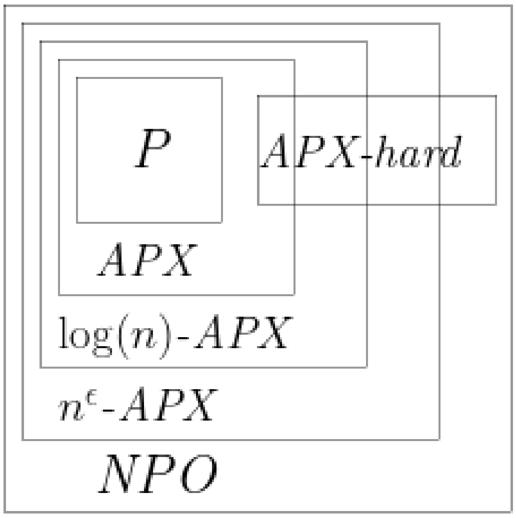


Figure 5.1: The diagram for classes of NPO.

- $g : I_A \times I_B \times \mathbb{Q} \cap (1, +\infty) \to I_A$, *for any* $x \in I_A$, *any rational* $\varepsilon \in (1, +\infty)$, *and any* $y \in sol_B(f(x, \varepsilon))$, *we have* $g(x, y, \varepsilon) \in sol_A(x)$ *and* g *can be computed in polynomial time with regard to both* $|x|$ *and* $|y|$.
- $h : (1, +\infty) \to (1, +\infty)$ *is computable and invertible.*
- *For any* $x \in I_A$, *any rational* $\varepsilon \in (1, +\infty)$, *and any* $y \in sol_B(f(x, \varepsilon))$, *if* $R_B(f(x, \varepsilon), y) \le h(\varepsilon)$, *then we have* $R_A(x, g(x, y, \varepsilon)) \le \varepsilon$.

Definition 5.20 [16] *Let A be an* NPO *problem, if for any other* NPO *problem B, B is PTAS-reducible to A, then A is said to be* APX-hard. *Further, if* $A \in$ APX *and* $A \in$ APX-hard, *A is said to be* APX-Complete.

Towards the approximability of hidden points and hidden vertices, the following arguments are already established in [19].

Theorem 5.21 [19] *Let A be the problem of maximum hidden vertex set in a simple polygon, then* $A \in$ APX-hard.

In the following chapters, we will see that it is indeed APX-Complete by finding a polynomial algorithm approximating it within a constant of 4.

Theorem 5.22 [19] *Let A be the problem of maximum hidden point set in a simple polygon, then there exists rational* $c > 1$ *such that A is not approximable within c as long as we assume* P $\neq$ NP.

Originally, in [19], it is reported to be APX-hard because they have different definitions. In our contexts, we can not have such statement as it is even not guaranteed to be an NPO problem.

5.4 Existential Theory of the Reals

In this section, we make a brief introduction to the existential theory of the reals hardness class $\exists\mathbb{R}$ and the semi-algebraic sets. This is a completely different scope other than the NP optimization compendium. Moreover, we prove that the decision version of the hidden point problem is in $\exists\mathbb{R}$.

First of all, we propose the decision problem of hidden point problem.

Definition 5.23 *We define the following language L such that for any polygon P and positive integer k, the tuple (P, k) is in L if and only if there exists an arrangement of k hidden points in polygon P.*

Given a polygon P and a positive integer k, the decision question of the hidden point problem, asks whether $(P, k) \in L$ or not.

It becomes oblivious that the hardness of the hidden point problem itself, denoted as S, is closely related to this decision problem, denoted as T. If T can be solved in polynomial time, so does S, as we can binary search the largest positive integer k such that the solution to T is positive. Similarly, if T is in NP, we know that S belongs to NPO. In another way, if we could solve S, we could also solve T easily since we already know the largest possible k.

By far, we still cannot conclude the hardness of T precisely. However, in this section, we will establish that T can be formulated as the feasibility of semi-algebraic sets, thereby placing it within the complexity class $\exists\mathbb{R}$.

5.4.1 Semi-algebraic Sets and $\exists\mathbb{R}$

The most fundamental object that algebraic geometry focuses on is the algebraic sets. In our contexts, we are dealing with semi-algebraic sets, which is induced by algebraic sets, via boolean combination of equalities and inequalities of real polynomials.

Definition 5.24 *Let I be a subset of the real vector space $\mathbb{R}^n$. I is called an algebraic set (algebraic variety) if there exists real polynomials $f_1, f_2, \cdots, f_k$ such that*

$$I = \{x \in \mathbb{R}^n | f_1(x) = f_2(x) = \cdots = f_k(x) = 0\}.$$

I is a basic semi-algebraic set if there exists real polynomials $f_1, f_2, \cdots, f_k$ and $h_1, h_2, \cdots, h_m$ such that

$$I = \{x \in \mathbb{R}^n | f_i(x) \geq 0, \forall i \in [k]; h_i(x) > 0, \forall i \in [m]\}.$$

I is a semi-algebraic set if it is the union of finite number of basic algebraic sets.

Further, we can consider the relationship between the class of semi-algebraic sets and formulas, and then define the existential theory of the reals.

Definition 5.25 *A quantifier-free formula on $\mathbb{R}^n$ is generated by following rules:*

- *Let $P : \mathbb{R}^n \to \mathbb{R}$ be a real polynomial, then $P * 0$ is a formula, where $* \in \{<, =, >\}$.*
- *If φ and ψ are both formulas, then their conjunction $\varphi \wedge \psi$, their disjunction $\varphi \wedge \psi$, and the negation $\neg\varphi$ are also formulas.*

Definition 5.26 *Let φ be a quantifier-free formula on n real variables $x_1, \cdots, x_n$, and $\psi = \exists x_1 \exists x_2 \cdots \exists x_n \varphi$. ψ belongs to the existential theory of the reals if it is true, and φ is said to be satisfiable in this case.*

We denote the set of all quantifier-free formulas as QFF. Via the scope of QFF, we can further define the complexity class $\exists\mathbb{R}$.

Definition 5.27 *Let A be a decision problems and I be the set of its instances. A is in $\exists\mathbb{R}$ if there exists a mapping $f : I \to QFF$, which is computable in polynomial time, such that for any instance $z \in I$, z is a yes instance if and only if the formula $f(z)$ is satisfiable.*

From the above definitions, one can see that for any semi-algebraic set S, there exists a quantifier-free formula φ such that $S = \{x|\varphi(x)\}$, as we can translate the intersection and union of the sets into the conjunction and disjunction of the formulas. Therefore, let P be the problem to decide whether a semi-algebraic set is empty or not, we can safely conclude that P is in the complexity class $\exists\mathbb{R}$.

Though a lot of efforts has been spent to formulate different problems in $\exists\mathbb{R}$, we still can not capture $\exists\mathbb{R}$ precisely in classical complexity classes. Up to now, there have been two milestones in attempting to locate $\exists\mathbb{R}$ in classical complexity hierarchy: [44] showed that it is NP-hard, and [12] proved that itself is contained in PSPACE. Since PSPACE $\subseteq$ EXPTIME, we have that problems in $\exists\mathbb{R}$ can be solved in exponential time. This is indeed non-trivial, as the best one prior to it is a double-exponential algorithm in [15].

5.4.2 Hidden Point Problem is in $\exists\mathbb{R}$

In this subsection, we will formulate the decision problem of hidden points as semi-algebraic sets, thus proving the following theorem.

Theorem 5.28 *Let $\mathcal{P}$ be the decision problem of hidden points, then $\mathcal{P} \in \exists\mathbb{R}$.*

Proof Let (P, m) be an instance of $\mathcal{P}$, where $P = (p_0, p_1, \cdots, p_{n-1})$ be a simple polygon and m is a positive integer. Let $w_{i,j}$ be the line passing vertex p_i and p_j and $W = \bigcup_{i \neq j} w_{i+j}$. Let $h_i = (x_i, y_i)$ be the i-th hidden point. We will construct a quantifier-formula φ over the coordinates $(x_i, y_i), i \in [m]$, such that P admit an arrangement of m hidden points if and only if φ is satisfiable.

Let $\mathcal{T} = (T_1, T_2, \cdots, T_{n-2})$ be a triangulation of P. If P admits m hidden points in it, by Corollary 5.4, we know that the feasible area of each point has non-empty interior, implying that it has positive Lebesgue measure. Therefore, each hidden point can be rearranged to another point in $P \setminus W$, as W only has zero Lebesgue measure. Therefore, we can always assume that $\forall i \in [m], h_i \in P \setminus W$, and it would never hurt the feasibility.

By definition, the formula φ can be decomposed as the following conjunction of sentences:

$$\varphi = \wedge_{i=1}^{m}[h_i \in P \setminus W] \wedge_{i \neq j} [\overline{h_i h_j} \not\subseteq P],$$

Let l_i be the sentence $[h_i \in P \setminus W]$ and $o_{i,j}$ be the sentence $[\overline{h_i h_j} \not\subseteq P]$. Since $h_i \in P \setminus W$, there exists exact one triangle $T_j \in \mathcal{T}$ such that $h_i \in int(T_j)$. Thus,

$$l_i = [\vee_{j=1}^{n-2}[h_i \in int(T_j)]] \wedge [h_i \notin W].$$

Let $s_{i,j}$ be the sentence $h_i \in int(T_j)$, and denote $T_j = (a_j, b_j, c_j)$. For any two distinct points u and v, let $H^+_{u,v}$ be the open halfplane which is on the left hand side of $\overrightarrow{uv}$. Then, $s_{i,j}$ can be further displayed.

$$s_{i,j} = [h_i \in H^+_{a_j,b_j}] \wedge [h_i \in H^+_{b_j,c_j}] \wedge [h_i \in H^+_{c_j,a_j}].$$

Let r_i be the sentence $[h_i \notin W]$, then s_i can be further decomposed as

$$r_i = \wedge_{j \neq k}[h_i \notin w_{j,k}].$$

Denote the sentence $[h_i \notin w_{j,k}]$ as $z_{i,j,k}$, then $z_{i,j,k}$ can be written as

$$z_{i,j,k} = [h_i \in H^+_{p_j,p_k}] \vee [h_i \in H^+_{p_k,p_j}]$$

Meanwhile, let's consider the sentence $o_{i,j}$. Since both h_i and h_j are in the interior of P, there exists an edge $e_k = (p_k, p_{k+1})$ of P such that e_k intersects with the segment $\overline{h_i h_j}$ properly. Therefore,

$$o_{i,j} = \vee_{k=0}^{n-1} [\overline{h_i h_j} \textit{ properly intersects with } \overline{p_k p_{k+1}}].$$

Suppose that for some specific k, $\overline{h_i h_j}$ properly intersects with the edge $\overline{p_k p_{k+1}}$, and let $t_{i,j,k}$ denote this sentence. Thus, these four points h_i, h_j, p_k, and p_{k+1} are in convex positions. Let $C = conv(\{h_i, h_j, p_k, p_{k+1}\})$ be their convex hull. Consider the order of vertices in C, there are two different cases: either $C = (h_i, p_k, h_j, p_{k+1})$ or $C = (h_i, p_{k+1}, h_j, p_k)$. Accordingly,

$$t_{i,j,k} = [C = (h_i, p_k, h_j, p_{k+1})] \vee [C = (h_i, p_{k+1}, h_j, p_k)].$$

Take $[C = (h_i, p_k, h_j, p_{k+1})]$ as an example, it can be formulated as the conjunction of 4 sentences.

$$\begin{aligned}[C = (h_i, p_k, h_j, p_{k+1})] = {} & [h_i \in H^+_{p_k, p_{k+1}}] \wedge [h_j \in H^+_{p_{k+1}, p_k}] \\ & \wedge [p_k \in H^+_{h_j, h_i}] \wedge [p_{k+1} \in H^+_{h_i, h_j}].\end{aligned}$$

Let $u = (x_u, y_u), v = (x_v, y_v), w = (x_w, y_w)$ be three points and $u \neq v$. As the final piece of the proof, we need to translate the formula with the form $w \in H^+_{u,v}$ into semi-algebraic sets. Indeed,

$$[w \in H^+_{u,v}] = [(x_w - x_u)(y_v - y_u) - (y_w - y_u)(x_v - x_u) < 0],$$

which only involves square-free quadratic polynomials.

Put all the pieces above together, we show that for each instance (P, m), we can construct a formula φ such that P admits m hidden points if and only if φ is satisfiable. Therefore, the decision problem of hidden points is in $\exists\mathbb{R}$. $\square$

Chapter 6

Spiral Polygon

We start our investigation on a series of polygon classes from the spiral polygon. It has the almost simplest structure, both in geometry and graph theory, such that it grants us the possibility to resolve the hard problems efficiently. Most importantly, it helps us to deduce some most basic and fundamental ideas in our contexts, including the convex/reflex chains and the continuous visibility graph.

Definition 6.1 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a simple polygon and p_0 is a convex vertex. P is called a spiral polygon if there exists $0 < l \leq r < n$ such that p_i is a reflex vertex if and only if $l \leq i < r$.*

Indeed, a spiral polygon is a polygon whose boundary is composed of two intervals whose vertices are all convex and reflex, respectively.

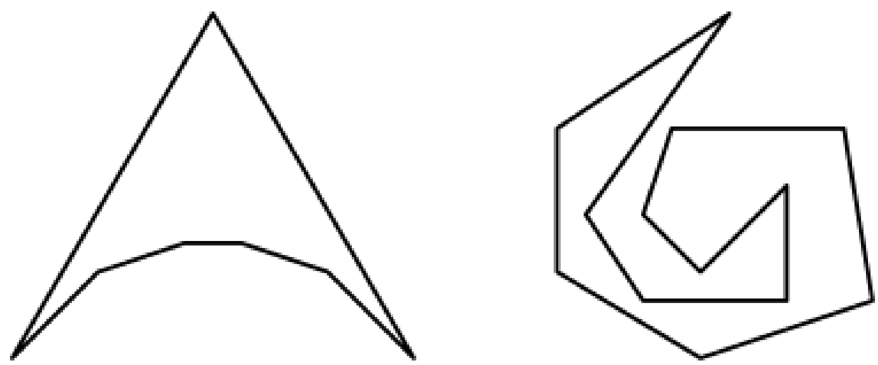


Figure 6.1: Figures of spiral polygons. The left one has only three convex vertices, which is the minimum for any simple polygon.

6.1 Convex Chain and Reflex Chain

In this section, we try to establish the geometric connections between the vertices of polygon, and further their relations to the maximum independent set and maximum clique, via the idea of the convex chain and the reflex chain.

Definition 6.2 *Let* $P = (p_0, p_1, \cdots, p_{n-1})$ *and* $L = (l_0, l_1, \cdots, l_{k-1}), k \geq 3$, *be a polygonal chain of the vertices of P in counter-clockwise order such that* $\forall i \in [0, k-2]$, *we have* $\overline{l_i l_{i+1}} \subseteq P$.

L is called a convex chain of P if for any $i \in [k-2]$, *vertex* l_{i+1} *lies on the* ***left*** *hand side of* $\overrightarrow{l_{i-1} l_i}$.

Respectively, L is called a reflex chain of P if for any $i \in [k-2]$, *vertex* l_{i+1} *lies on the* ***right*** *hand side of* $\overrightarrow{l_{i-1} l_i}$. [1]

The vertices are in the counter-clockwise order means that we can start at l_0, walk through ∂P, and visit the vertices of L in order without visiting any vertex twice. Meanwhile , traverse through a convex/reflex chain takes a left/right turn on each intermediate vertex. See the following figure for illustration.

Before we propose further arguments about the convex and reflex chains, we present the following topology fact without a proof, based on which we continue our discussion.

Lemma 6.3 Intersecting Chords in Jordan Curve *Let J be a Jordan curve, and R be the closed region enclosed by J. Let* $\{a, b, c, d\} \subseteq J$, *and* $e_1 = (a, b), e_2 = (c, d)$ *be chords of J such that* $e_1 \subseteq R, e_2 \subseteq R$, *and* e_1 *properly intersects with* e_2.

Note that a and b divide J into two parts, then we have c and d belong to different subdivisions of J.

Proposition 6.4 *Let P be a simple polygon, and L be a convex chain (or a reflex chain) of P, then L is a polygonal chain without self-intersection.*

Proof Let $L = (v_0, v_1, \cdots, v_{k-1})$ and $e_i = (v_i, v_{i+1}), e_j = (v_j, v_{j+1})$ be two edges in L that are not adjacent. By the definition, v_i, v_{i+1}, v_j and v_{j+1} are in counter-clockwise order. Note that v_i and v_{i+1} divides ∂P into two halves J_1 and J_2. Since v_j and v_{j+1} are in the same half, by Lemma 6.3, we can see that e_i can not intersect with e_j. □

[1] By our assumption, any three vertices in P can not be collinear, so travelling through such a polygonal chain must take a left turn or a right turn at each stop.

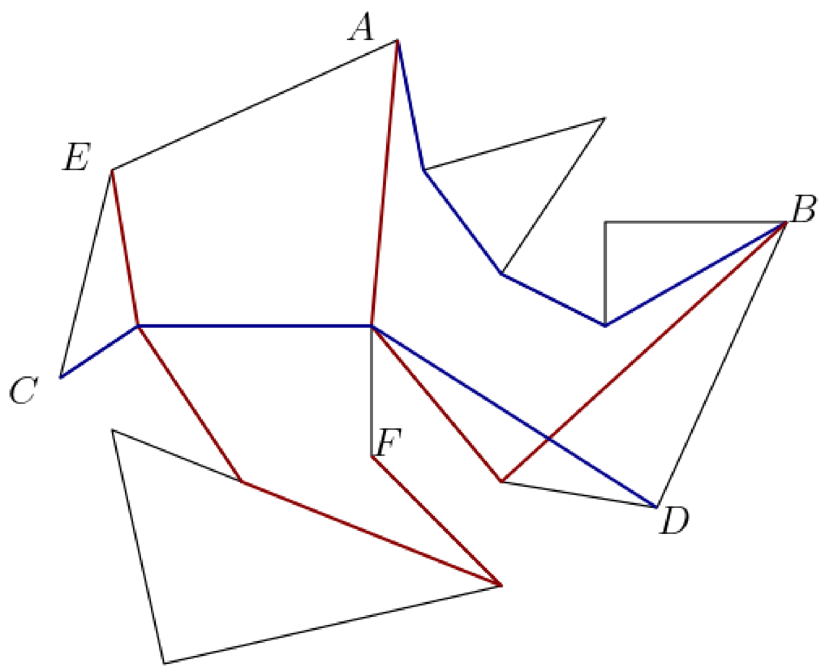


Figure 6.2: The red and blue polygonal chains are convex and reflex chains respectively. Notice that the edges of these chains are not necessarily the edge of the polygon, and there are convex chain from A to B and reflex chain from B to A at the same time.

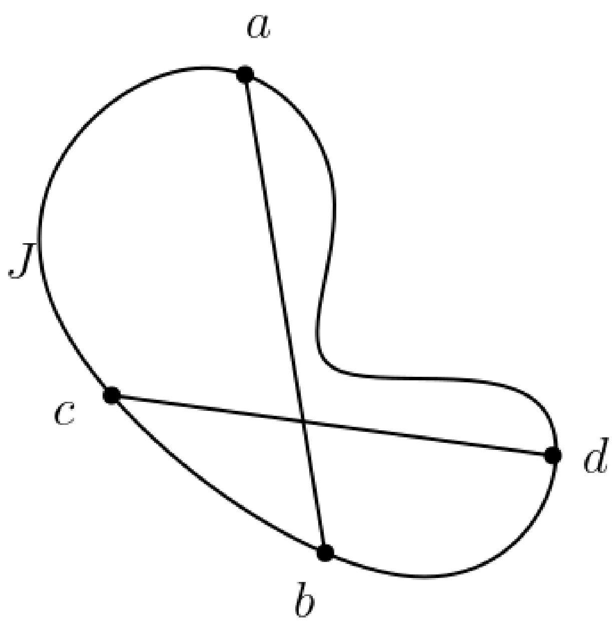


Figure 6.3: This figures illustrates Lemma 6.3.

If we take a look at the generic shape of a convex chain or a reflex chain, we would find that it ensembles a two dimensional spiral, which is a curve spinning around the origin with increasing distance. In fact, we present the following fact without proving it.

Lemma 6.5 *Let P be a simple polygon, and $C = (v_0, v_1, \cdots, v_{k-1})$ be a convex chain (or a reflex chain) in P. Denote $conv(C)$ as Q. Then, there exists $0 \leq l \leq r \leq k$, such that*

- $\forall l \leq i < r, v_i \in \partial C$.
- $\forall 0 \leq i < l$ *or* $r \leq i < k$, $v_i \in int(C)$.

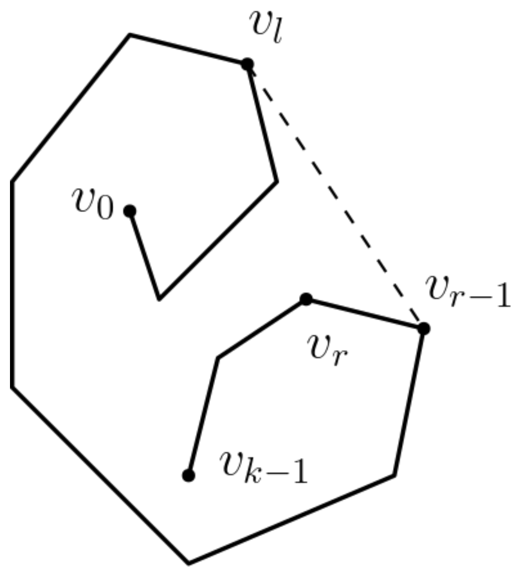


Figure 6.4: This figure illustrates Lemma 6.5, and also the generic shape of a convex chain. Here one can see that (v_l, v_{r-1}) is an edge of the convex hull

.

6.1.1 Reflex Chain

Proposition 6.6 *Let P be a simple polygon and (u, v, w) be a reflex chain in P, then vertex u and w are invisible to each other.*

Proof We will argue that $\overline{uw} \not\subseteq P$ by finding a point $r \in int(\overline{uw})$ such that $r \notin P$.

Notice that P is a simply connected subspace and $v \in \partial P$, then there exists a simple plane curve C from point v to the point of infinity such that $C \cap P = \{v\}$. Let $Q = P \cup T$, where $T = (u, w, v)$ is the triangle with vertices u, v and w.

Since (u, v, w) is a reflex chain, the reflex angle at vertex v is included in P, and the convex angle at v is included T, we have $v \in int(Q)$. Therefore, there exists $\varepsilon > 0$ such that the l_2-ball $B(v, \varepsilon) \subseteq Q$. However, as $v \in \partial P$, for any $\varepsilon > 0$, we have $B(v, \varepsilon) \nsubseteq P$. Accordingly, there exists point q such that $q \in B(v, \varepsilon), q \notin P$, and $q \in T$.

Notice that $\overline{uv} \subseteq P$ and $\overline{vw} \subseteq P$, then we have $q \notin \overline{uv}$ and $q \notin \overline{vw}$. Suppose that $q \in \overline{uw}$, then we already get the wanted point r. Otherwise, we have $u \notin \overline{uw}$ and $q \in int(T)$.

Further, consider the curve $D \subseteq C$ which connects the point q and the point of infinity. Since ∂T is a Jordan curve and q is in the interior of T, we have $D \cap \partial T \neq \varnothing$. Suppose point $r \in D \cap \partial T$, then we have $r \in int(\overline{uw})$. Hence, since $D \cap P = \varnothing$, we have $r \notin P$. Thus, $\overline{uw} \nsubseteq P$, and vertices u and w are invisible to each other. □

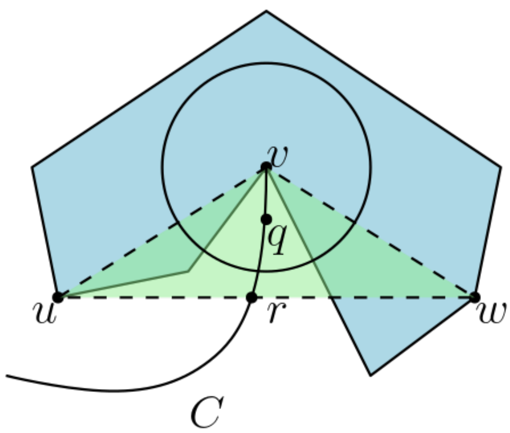


Figure 6.5: This illustrates the proof of Proposition 6.6. The green area indicates T which the blue area indicates P.

Lemma 6.7 *Let P be a simple polygon and $L = (v_0, v_1, \cdots, v_{k-1})$ be a reflex chain on $k \geq 3$ vertices in P. Let $v_i, v_j, i < j$ be two vertices in L. then v_i and v_j are visible to each other if and only if $j = i + 1$. In other words, vertices are pairwise invisible unless they are neighbours in the reflex chain L.*

Proof We prove this by induction. For the case $k = 3$, it is induced by Proposition 6.6. Suppose that it holds for $k \leq m$, and consider the case when $k = m + 1$.

Let $k = m + 1$ and $L = (v_0, v_1, \cdots, v_m)$ be the reflex chain. We need to argue that v_0 is indeed invisible to v_m. Suppose the opposite that v_0 and v_m are visible to each other, then L can be extended to an ordered cycle $C = (v_0, v_1, \cdots, v_m, v_0)$ of $m + 1$ vertices. By Lemma 2.25, L has at least $m - 2 \geq 1$ chords. Thus, there exists $v_i, v_j, 1 < i - j < m$ such that $(v_i, v_j) \in VG(P)$. However, by the induction, $(v_i, v_{i+1}, \cdots, v_j)$ is a reflex chain of $j - i + 1$ vertices, implying that v_i is actually invisible to v_j and giving rise to a contradiction.

Therefore, for $k = m + 1$, the statement holds as well. □

Corollary 6.8 *Let P be a simple polygon and $L = (v_0, v_1, \cdots, v_{k-1})$ be a reflex chain on $k \geq 3$ vertices in P, then there exists a hidden vertex set $H \subseteq V(L)$ such that $|H| = \lceil \frac{k}{2} \rceil$.*

Proof Let $t = \lceil \frac{k}{2} \rceil$, and $H = \{v_0, v_2, \cdots, v_{2(t-1)}\} \subseteq V(L)$. By Lemma 6.7, H is a hidden vertex set and $|H| = t = \lceil \frac{k}{2} \rceil$. □

Lemma 6.9 *Let P be simple polygon whose vertices are in general position and C be a clique in $VG(P)$, then vertices in C are in convex position.*

Proof For the case $|C| \leq 3$, as the vertices are in general position, they are always in convex position.

Consider the case $|C| \geq 4$. Suppose the opposite that vertices in C are not in convex position. Then, there exist four vertices $\{v_a, v_b, v_c, v_d\}$ such that $v_a \in int(conv(\{v_b, v_c, v_d\}))$, where v_b, v_c, and v_d are in counter-clockwise order in ∂P. Notice that these three vertices subdivide ∂P into three subchains, and v_a belongs to exact one of them. Suppose that $v_a \in P(v_b, v_c)$, then we have (v_b, v_a, v_c) is a reflex chain in P. This means that v_b is actually invisible to v_c, contradicting to that C is a clique.

In summary, vertices in C are always in convex position. □

Although the class of visibility graph is already so special, if we try to solve most of the graph optimization problems in it, we would still find that these hard problems remains quite hard. For example, the maximum independent set, minimum clique cover, and the minimum dominating set are still NP-hard in visibility graphs of simple polygon [34] . In spite of that, Lemma 6.9 grants us a chance to efficiently find the maximum clique in the visibility graph of a simple polygon. To find a clique, we are actually looking for a convex hull in the polygon, and its geometric structure is easy to characterize [29].

6.1.2 Convex Chain

In last subsection, we have seen that the presence of the reflex chain implies the existence of a hidden vertex set. In this subsection, we will see that rather than the independent set, the convex chain is likely to be associated with cliques in the visibility graph.

Proposition 6.10 *Let P be a simple polygon and $L = (v_0, v_1, \cdots, v_{k-1})$ be a convex chain in P. If v_0 is visible to v_{k-1}, then there exists $C \subseteq V(L)$ such that $|C| \geq \lceil \frac{k}{3} \rceil$ and vertices in C are pairwisely visible to each other.*

Proof Since $\overline{v_0 v_{k-1}}$ is a chord of ∂P and $\overline{v_0 v_{k-1}} \subseteq P$, by Lemma 6.3, $\overline{v_0 v_{k-1}}$ does not intersect with L except for the vertex v_0 and v_{k-1}. Therefore, $Q = (v_0, v_1, \cdots, v_{k-1})$ is indeed a simple polygon, and $Q \subseteq P$.

Hence, notice that for all $i \in [k-2]$, v_i is a convex vertex in Q. Thus, Q has at most 2 reflex vertices, which might be v_0 and v_{k-1}. By Lemma 3.12, we have $\textsc{Cover}(Q) \leq 3$. Further, let $\{C_1, C_2, C_3\}$ be a convex covering of Q, and denote $X_i = V(L) \cap C_i$. Then, since $\bigcup_{i=1}^{3} X_i = V(L)$, there exists $k \in [3]$ such that $|X_k| \geq \lceil \frac{k}{3} \rceil$. Notice that vertices in X_k are pairwisely visible, and X_k is indeed the wanted clique. □

Via Proposition 6.10, we can further see that a convex chain indeed implies the presence of a clique or an independent set in the visibility graph by the following lemma.

Lemma 6.11 *Let P be a simple polygon and $L = (v_0, v_1, \cdots, v_{k-1})$ be a convex chain in P, then there exists $X \subseteq V(L)$ and $|X| \geq \lfloor \sqrt{\frac{k}{3}} \rfloor$ such that X is an independent set or a clique in $VG(P)$.*

Proof Let $t = \lfloor \sqrt{3k} \rfloor$.

Suppose that there exists v_i, v_j such that $i + t \leq j$ and v_i is visible to v_{j-1}. Then, consider the convex chain $L' = (v_i, v_{i+1}, \cdots, v_{j-1})$ which have at least t vertices. By Proposition 6.10, we know that there exists $X \subseteq V(L')$ such that $|X| \geq \frac{t}{3} \geq \lfloor \sqrt{\frac{k}{3}} \rfloor$.

Otherwise, suppose that for all v_i, v_j with $i + t \leq j$, v_i is invisible to v_{j-1}. Now, let's consider the vertex set $X = \{v_0, v_t, \cdots, v_{mt}\}$ where $m = \lfloor \frac{k-1}{t} \rfloor$.

By the assumption, vertices in X are pairwisely invisible. So X is indeed an independent set in $VG(P)$, and $|X| \geq \lceil \frac{k}{t} \rceil \geq \lfloor \sqrt{\frac{k}{3}} \rfloor$. □

A graph is called k-Ramsey graph if both its maximum clique and maximum independent set have size at most k. Determining the smallest value of k with respect to the number of vertices n such that a k-Ramsey graph exists is one of the most well-known questions in Ramsey theory. In general graph, the answer is almost clear. A Probabilistic method can show us that there always exists a $O(\log n)$-Ramsey graph [21]. Meanwhile, when it comes to the construction, the $(\log n)^{O(1)}$-Ramsey graph can be realized explicitly [33], though it does not match the theoretical optimal.

However, when considering this question this question in visibility graph, nothing has been known yet. Since the visibility graph is a special class of graph, it is unlikely that this question yields the same answer as in general graphs. Lemma 6.11 indicates that in a convex chain of k vertices, we can always find either a clique or an independent set of $\Omega(\sqrt{k})$ size. Hence, Corollary 6.8 shows that a reflex chain ensures an independent set of linear size.

Although these observations are not sufficient yet to provide a solution to this question, we still hold the belief that for k-Ramsey visibility graphs, k should have asymptotic lower bound significantly higher than $\log n$. Thus, we propose the following conjecture.

Conjecture 6.12 *There exist constants $\varepsilon > 0, c > 0$ such that for any simple polygon P with n vertices, $max\{\alpha(VG(P)), \omega(VG(P))\} \geq cn^{\varepsilon}$.*

6.2 Spiral Polygon

By the definition of convex and reflex chains, we can see that the boundary of a spiral polygon can be decomposed into a convex chain and a reflex chain that are disjoint.

Definition 6.13 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a spiral polygon such that $\forall i \in [0, k-1]$, p_i is convex, and otherwise p_i is reflex. Then, the convex interval of R is defined as $L(P) := (v_0, v_1, \cdots, v_{k-1})$, which is a convex chain. Similarly, the reflex interval of P is defined as $R(P) = (v_{k-1}, v_k, \cdots, v_{n-1}, v_0)$, which is a reflex chain.*

Indeed, spiral polygons possess a specialized geometric structure, providing simple solutions to otherwise hard geometry problems, including the hidden points and hidden vertices problem.

Lemma 6.14 *Let P be a spiral polygon with r reflex vertices, then* $\mathrm{HP}(P) = \textsc{Cover}(P) = r + 1$.

Proof It suffices to argue that there exists a hidden point set of size $r+1$ in P, as by Lemma 3.12 we know that $\mathrm{HP}(P) \leq \textsc{Cover}(P) \leq r+1$.

Let $R = (v_0, v_1, \cdots, v_{r+1})$ be the reflex interval of P. For $i \in [0, r]$, let u_i the midpoint of $\overline{v_i v_{i+1}}$. Then, for any $i < j$, $(u_i, v_{i+1}, \cdots, v_j, u_j)$ is a reflex chain, if we consider both u_i and u_j as vertices of P. By Lemma 6.7, u_i is invisible to u_j. Therefore, $X = \{u_0, u_1, \cdots, u_r\}$ is indeed a hidden point set with $r+1$ points. □

Lemma 6.15 *Let P be a spiral polygon, then $VG(P)$ is an interval graph.*

Proof This is proved in [24]. □

Corollary 6.16 *Let P be a spiral polygon of n vertices and G be its visibility graph, then* $\max\{\alpha(G), \omega(G)\} \geq \sqrt{n}$.

Proof By Lemma 6.15, G is an interval graph. Thus, G is perfect, and its chromatic number is the same as the clique number, $\chi(G) = \omega(G)$. Hence, we know that $\chi(G)\alpha(G) \geq n$, as a proper coloring of G is a partition of the vertices into independent sets, where each of them has at most $\alpha(G)$ vertices. Therefore, we have $\alpha(G)\omega(G) \geq n$, and $\max\{\alpha(G), \omega(G)\} \geq \frac{\alpha(G)+\omega(G)}{2} \geq \sqrt{\alpha(G)\omega(G)} \geq \sqrt{n}$. □

This corollary shows that the visibility graph of a spiral polygon can be a $k-$Ramsey graph only if $k \geq \sqrt{n}$, which is much larger than the $O(\log n)$ bound in general graphs.

Lemma 6.17 *Let P be a spiral polygon with r reflex vertices, then* $\lceil \frac{r}{2} \rceil + 1 \leq \mathrm{HV}(P) \leq r + 1$.

Proof The upper bound is induced by Lemma 3.12, and the lower bound is implied by Corollary 6.8 because the reflex interval of P is a reflex chain with $r+2$ vertices. □

6.2.1 Continuous Visibility Graph

In last subsection, we showed that in a spiral polygon, the maximum hidden point set and minimum convex covering shares the same size.

Then, here comes the question: does this take place in every simple polygon? The answer is certainly not. The pentagram is a cute counter-example to that, as is illustrated in the following figure.

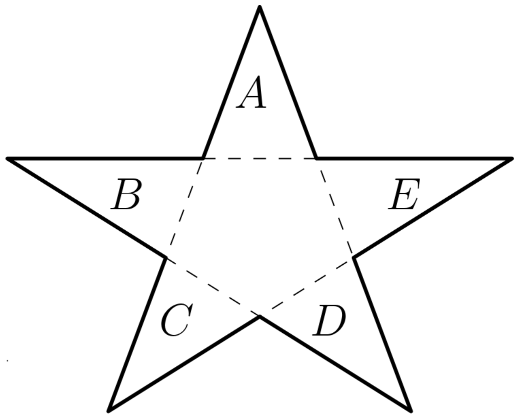


Figure 6.6: This is an example shows why maximum hidden point set and minimum convex covering shares the same size do not always have the save size.

The above pentagram is partitioned into six convex pieces, including five triangles, denoted as $\{A, B, C, D, E\}$, and a pentagon. If we try to place a hidden point u in the A, then every point in C and D is visible to u. Hence, notice that $B \cup E$ can only host one extra hidden point, we have the hidden point set has size at most 2. Meanwhile, consider the induced visibility graph on five convex vertices, denoted as G'. It is clear that G' is triangle-free, thus each convex piece can only cover two of them. Therefore, the minimum convex covering is at least 3.

However, we are not satisfied just by such a special case. It is still interesting that in which class of simply polygons, the minimum convex covering has the same size as the maximum hidden point set. This is where we need to introduce the idea of continuous visibility graph.

The traditional vertex visibility graph, can only be used to characterized the problem in a discrete space, for example, the hidden vertex set. However, when it comes to the problem whose feasible solution resides in a continuous space, it becomes less utilized. We propose the continuous visibility graph, and we believe that it is helpful to address this issue.

Definition 6.18 *Let P be a simple polygon, the continuous visibility graph of P, denoted as $CVG(P)$, is defined as $G = (V, E)$, where $V = P$, and for any $u \in P, v \in P, u \neq v$, we have $(u, v) \in E$ if and only if $\overline{uv} \subseteq P$.*

Different from the visibility graph, the continuous visibility graph itself has all the points in the polygon as its vertex set, which means that it is indeed an infinite graph. But, the following theorem can help us migrate our experience from finite graphs to infinite graphs.

Theorem 6.19 De Bruijn–Erdős theorem [11]*: Let G be a infinite graph, if any finite subgraph of G is c-colorable, then G is also c-colorable.*

In other words, $\chi(G) = \max\{\chi(G')|G' \subseteq G, G' \textit{ is finite}\}$.

Remark 6.20 *The theorem here depends on the adoption of axiom of choice. Hence, if there is no finite supremum on the chromatic number of finite subgraphs, then G has infinite chromatic number.*

Corollary 6.21 *Let G be a infinite graph, if any finite subgraph of G can be covered by c cliques, then G itself can be covered by c cliques as well.*

In other words, $\kappa(G) = \max\{\kappa(G')|G' \subseteq G, G' \textit{ is finite}\}$.

Proof This is implied by the De Bruijn–Erdős theorem as the a proper coloring in G is indeed a clique cover in $\overline{G}$. □

Then, we can discuss the hidden points and convex covering in simple polygon, with the language of continuous visibility graph.

Lemma 6.22 *Let P be a simple polygon, and* $G = CVG(P)$ *be its continuous visibility graph, then* $\mathrm{HP}(P) = \alpha(G)$.

This is indeed the translation of the definition.

Lemma 6.23 *Let P be a simple polygon, and* $G = CVG(P)$ *be its continuous visibility graph, then* $\textsc{Cover}(P) = \kappa(G)$.

Proof It is clear that $\kappa(G) \leq \textsc{Cover}(P)$. Since any convex shape in P is a clique in G, any convex covering of P is also a proper clique cover of G.

Further, let $k = \kappa(G)$, and $\{C_1, \cdots, C_k\}$ be the minimum clique cover of G. Then, $\forall i \in [k]$, because C_i is a clique in $CVG(P)$, analogous to Lemma 6.9 [2], we have $conv(C_i) \subseteq P$. Therefore, $\mathcal{C} = \{conv(C_1), \cdots, conv(C_k)\}$ is a convex covering of P, and $\textsc{Cover}(P) \leq \kappa(G)$.

Therefore, $\textsc{Cover}(P) = \kappa(G)$. □

[2]We can not generalize the arguments in finite number of points simply. The convex hull of infinite number of points might no longer be a convex polytope. For example, the convex hull of a circle is a disk, which has infinite number of extremal points.

Then, we can conclude that the reason why $\mathrm{HP}(P) = \textsc{Cover}(P)$ in spiral polygon P is that the maximum independent set has then same size as minimum clique cover in $CVG(P)$. Indeed, we can further show that the continuous visibility graph of the spiral polygon is a chordal graph.

Definition 6.24 *Let G be a graph, G is called a chordal graph if any cycle with $k \geq 4$ vertices in G has at least a chord.*

Definition 6.25 *Let P be a simple polygon, and u, v be two distinct points in P. The geodesic path between u and v is defined as the shortest curve whose endpoints are u and v.*

Remark 6.26 *In a simple polygon, the geodesic path between any two points should always be a polygonal chain. [39]*

Theorem 6.27 *Let P be a spiral polygon and G be its continuous visibility graph, then G is a chordal graph.*

Proof Let $C = (v_0, v_1, \cdots, v_{k-1}), k \geq 4$ be a cycle in G. There are two different scenarios to consider.

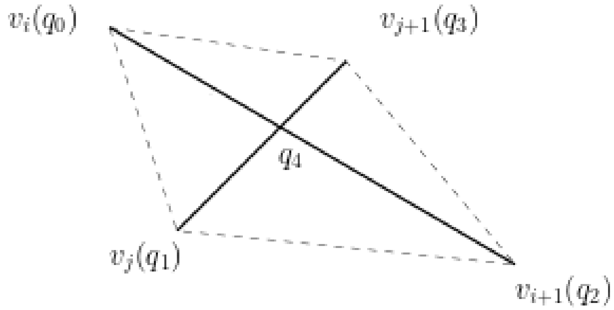
Case 1: The closed polygonal chain $(v_0, v_1, \cdots, v_{k-1})$ is not self-intersecting, then $Q = (v_0, v_1, \cdots, v_{k-1})$ is a simple polygon and $C = \partial Q$. Therefore, by the necessary condition of visibility graph, the cycle $(v_0, v_1, \cdots, v_{k-1})$ has at least $k-3$ chord in $VG(Q)$. Denote any one of them as (u, v). Since $Q \subseteq P$, we have $\overline{uv} \subseteq P$, and (u, v) is also a chord of the cycle C.

Case 2: The closed polygon chain $(v_0, v_1, \cdots, v_{k-1})$ is self-intersecting. Suppose the edge $e_i = (v_i, v_{i+1})$ intersects with the edge $e_j = (v_j, v_{j+1})$, and they are not adjacent. Further, we could assume that these four vertices are in convex position. Otherwise, if any three of them are collinear, we would have a triangle in G, and at least one edge of it is a chord of C.

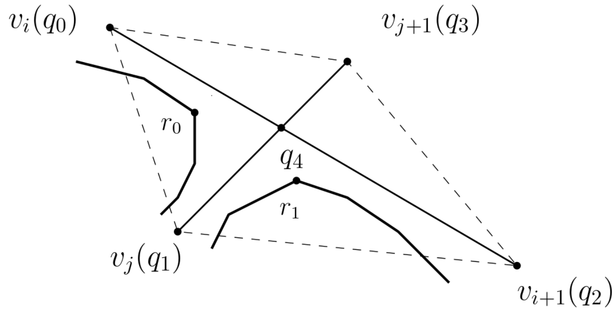
Thus, we assume that these four vertices are in convex position and e_i properly intersects with e_j, as illustrated in the following figure.

Let $Q = conv\{v_i, v_{i+1}, v_j, v_{j+1}\}$, and rewrite it as $Q = (q_0, q_1, q_2, q_3)$, and q_4 be the intersection of the diagonals. Then, we are about to argue that at least three edges of Q are in G.

Suppose the opposite, which mean that at least two edges of Q are absent in G. Without loss of generality, assume that $(q_0, q_1) \notin G$ and $(q_1, q_2) \notin G$.



For any two distinct points u, v in P, let $l(u, v)$ be the geodesic path connecting u and v, then we have all the intermediate vertices of $l(u, v)$ are reflex vertices of P. By our earlier assumption, $\overline{q_0 q_1} \not\subseteq P$, the geodesic path $l(q_0, q_1)$ is not a segment. Therefore, there exists a reflex vertex r_0 such that $r_0 \in int((q_0, q_1, q_4))$. Similarly, there exists another reflex vertex r_1 such that $r_1 \in int((q_1, q_2, q_4))$, which is shown in the following figure.



However, notice that r_0 and r_1 lie on the different sides of $\overline{q_1 q_3}$, it is impossible to connect r_1 and r_1 via a reflex chain without intersecting with the segment $\overline{q_1 q_3}$.

Thus, at least three edges in $E(Q)$ exist in G. Consider the induced graph $G[V(Q)]$, there are four vertices but at least five edges. So at least one edge in this induced graph is a chord of C.

In both scenarios, we have shown that every cycle of at least four vertices has a chord. Therefore, G is indeed a chordal graph. $\square$

This theorem tells us that G is a perfect graph. Thus, $\alpha(G) = \kappa(G)$, and $\mathrm{HP}(P) = \textsc{Cover}(P)$.

Chapter 7

2-Convex Polygon

Let P be a simple polygon, it is already well-known that HP(P) $\leq$ Cover(P). Of course, HP(P) does not necessarily equals to Cover(P), then it becomes interesting to ponder the potential gap between them. In other words, let P be a simple polygon on n vertices, and $G = CVG(P)$ be its continuous visibility graph, we inquire about the ratio $\kappa(G)/\alpha(G)$ with regard to n?

A recent breakthrough in [9] shows that $\kappa(G)/\alpha(G) \leq 8$, giving the first constant upper bound. However, the largest ratio $\kappa(G)/\alpha(G)$ known to us is much less than 8, suggesting that this upper bound might not be tight and could be further improved.

In this chapter, we will see that if $\alpha(G) \leq 2$, the ratio is bounded by $3/2$.

7.1 k-Convex Polygon

Before we step into the scope of other polygons which is more complicated, we would like to discuss the k-convexity first, the characterization of which is previously discussed in [6]. Later, we would talk about funnel polygons and pseudotriangles in next chapter, which are indeed special cases of 2-convex polygons.

Definition 7.1 *Let M be a collections of finite number of compact sets in the plane, M is said to be k-convex if for any straight line l in the plane, $M \cap l$ has at most k connected components.*

Under such condition, M is called a k-convex set, and a simple polygon P is called k-convex polygon if itself is k-convex.

Indeed, the k-convexity is a generalization of the convexity. That is, if M is 1-convex, it is exactly convex as well. Meanwhile, we can see that a k-convex set is allowed to be not connected. For example, the set of two disjoint convex polytope is consider to 2-convex. Moreover, we can see that this general convexity is somehow preserved through the set operation.

Lemma 7.2 *Let P be a k_1-convex polygon and Q be a k_2-convex polygon, then $P \cup Q$ is $(k_1 + k_2)$-convex and $P \cap Q$ is $(k_1 + k_2 - 1)$-convex.*

Proof Denote $P \cup Q$ as X and $P \cap Q$ as Y, and let l be a straight line in the plane.

$P \cup Q$: $l \cap X = (l \cap P) \cup (l \cap Q)$. Notice that $l \cap P$ and $l \cap Q$ have at most k_1 and k_2 connected components respectively. Then, X has at most $k_1 + k_2$ connected components, and $P \cup Q$ is $(k_1 + k_2)$-convex.

$P \cap Q$: Suppose that $l \cap P$ has k_1 connected components and $l \cap Q$ has k_2 connected components. Otherwise, we have $l \cap Y \subseteq l \cap X$, and the later one has less than $k_1 + k_2$ connected components, implying that $l \cap Y$ has less than $k_1 + k_2$ connected components.

Let $\Gamma : l \to \mathbb{R}$ be a function such that

$$\forall u, v \in l, \|u - v\|_2 = |\Gamma(u) - \Gamma(v)|.$$

Let $S = \Gamma(l \cap P)$ and $T = \Gamma(l \cap Q)$. Then S is a collection of k_1 disjoint connected compact sets, and T is a collection of k_2 disjoint connected compact sets.

Since any compact connected set in a line is either a point or a segment, S can be formulated as $S = \bigcup_{i=1}^{k_1} [a_i, b_i]$, and T can be formulated as $T = \bigcup_{i=1}^{k_2} [c_i, d_i]$, where $\forall i$, $a_i \leq b_i < a_{i+1}$, and $c_i \leq d_i < c_{i+1}$.

Then we need to argue that $S \cap T$ has at most $k_1 + k_2 - 1$ connected components, and we prove this by induction on $k_1 + k_2$.

For the case $k_1 + k_2 = 2$, $k_1 = k_2 = 1$, we clearly have $S \cap T$ has at most 1 connected components.

Suppose for all the cases with $k_1 + k_2 < n$ the proposition stands, and consider the case $k_1 + k_2 = n$. Without loss of generality, suppose that $b_1 \leq d_1$. Thus, $S \cap T = ((S \setminus [a_1, b_1]) \cap T) \cup ([a_1, b_1] \cap T)$.

By induction, $(S \setminus [a_1, b_1]) \cap T)$ has at most $(k_1 - 1) + k_2 - 1 = k_1 + k_2 - 2$ connected components. Hence, notice that for $i \geq 2$, $[a_1, b_1] \cap [c_i, d_i] = \varnothing$, thus $[a_1, b_1] \cap T$ has at most 1 connected components. Therefore, $S \cap T$ has at most $k_1 + k_2 - 1$ connected component, and this completes the induction.

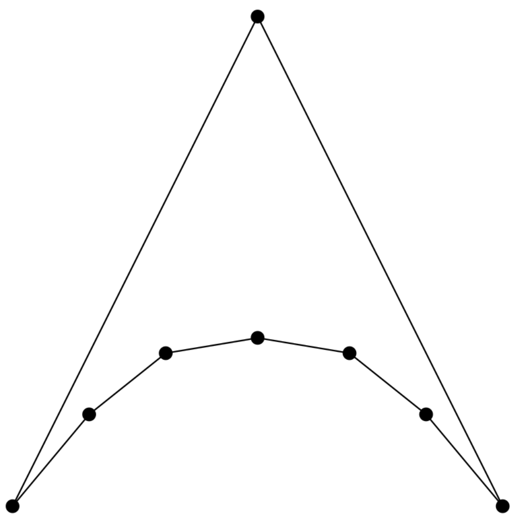
Therefore, we conclude that $P \cap Q$ is indeed $(k_1 + k_2 - 1)$-convex. □

Then, we can easily establish the connection between the k-convexity and the hidden point number.

Lemma 7.3 *Let P be a simple polygon and* $\mathrm{HP}(P) = k$*, then P is a k-convex polygon.*

Proof Let l be a line in the plane, and suppose that $P \cap l$ has m connected components, denote them as $C_1, C_2, \cdots, C_m$. For each C_i, let u_i be a point in C_i. Notice that for any $i \neq j$, $\overline{u_i u_j} \nsubseteq P \cap l$, otherwise they are in the same connected component. Hence, since $\overline{u_i u_j}$ is always in the subspace l, we have $\overline{u_i u_j} \nsubseteq P$. Then, $X = \{u_1, u_2, \cdots, u_m\}$ is indeed a hidden point set in P. Note that $\mathrm{HP}(P) = k$, we have $m \leq k$, and P is a k-convex polygon. □

In another way, given a k-convex polygon P, can we guarantee an upper bound for $\mathrm{HP}(P)$ with regard to k? The answer is actually negative. Even if P is 2-convex, it can still admit an arbitrary large hidden point set. See the following figure for an illustration.



The above figure shows a spiral polygon P with only three convex vertices, which is arranged consecutively on the boundary. It is clear that P is indeed 2-convex, and $\mathrm{HP}(P) = r + 1$, where r is the number of reflex vertices in P. Therefore, an arbitrary large hidden point set can be located in P as long as we arrange more reflex vertices there.

7.2 2-Convex Polygon

Among the class of k-convex polygons, the 2-convex polygon draws the most attention from us, as they have the simplest characterization of the structure (apart from the convex polygon, of course).

Lemma 7.4 *Let P be a 2-convex polygon, and u, v be any two consecutive vertices in the convex hull of P. Let $C = P(u, v) = (q_0, q_1, \cdots, q_{t-1})$ be the polygonal chain in P connecting vertex u and v in counterclockwise order, where $q_0 = u$ and $q_{t-1} = v$. Then, there exists $0 \le r \le s < t$ such that*

- *$\forall i \in [0, r), q_i$ is convex in P.*
- *$\forall i \in [r, s), q_i$ is reflex in P.*
- *$\forall i \in [s, t), q_i$ is convex in P.*

Proof This is proved by Lemma 12 in [6]. $\square$

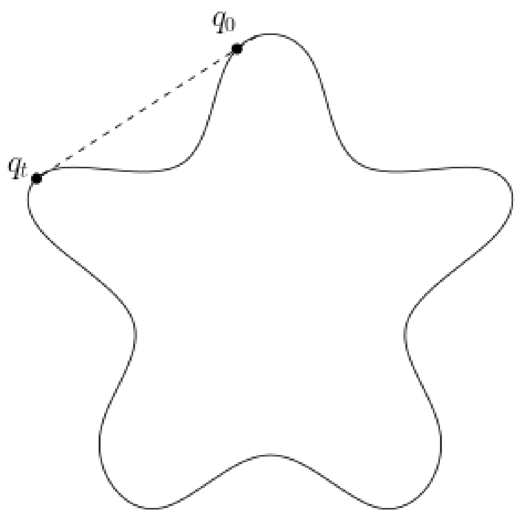


Figure 7.1: q_0 and q_t are consecutive vertices in the convex hull of P.

In a 2-convex polygon P, let u, v be two consecutive vertices in the convex hull of P, which are not adjacent in P. Then, the boundary of P between u and v can be decomposed into at most two convex chains and a reflex chain. According to that, the following figure illustrates the generic shape of a 2-convex polygon.

Definition 7.5 *Let C be a polygonal chain, and l be a straight line in the plane. The crossing number of C and l, denoted as $cr(C, l)$, is defined as the number of connected components of $C \cap l$.*

The following figure shows the different situations of counting the crossing numbers.

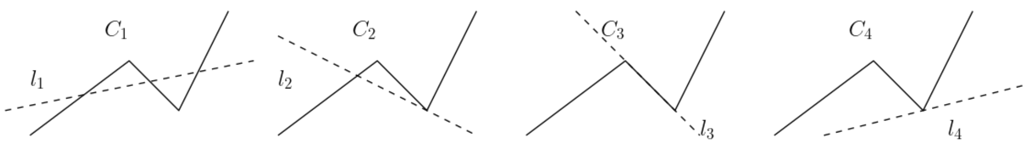


Figure 7.2: The figures illustrates the crossing number between identical polygonal chains and different lines, where $cr(C_1, l_1) = 3, cr(C_2, l_2) = 2, cr(C_3, l_3) = 1$, and $cr(C_4, l_4) = 1$.

Lemma 7.6 *Let P be a 2-convex polygon, and C be a reflex chain in P. Then for any line l in the plane, we have $cr(C, l) \leq 2$.*

Proof We prove this by contradiction. Suppose that $cr(C, l) \geq 3$, and let v_1, v_2 and v_3 be three points in different connected components of $C \cap l$, and v_1, v_2, v_3 are in order with regard to l.

Let $V = \{v_1, v_2, v_3\}$. By the assumption, we know that for any $i \neq j$, v_i and v_j are not in the same edge of C, otherwise they should be in the same connected component of $C \cap l$. Thus, by Lemma 6.7, points in V are pairwisely invisible. Thus, there exists $u_1 \in int(\overline{v_1 v_2}), u_2 \in int(\overline{v_2 v_3})$, such that $u_1 \notin P$ and $u_2 \notin P$. Therefore, we can see that points in V also belong to different connected components of $P \cap l$. This means that $P \cap l$ has at least 3 connected components, contradicting to the assumption that P is 2-convex. $\square$

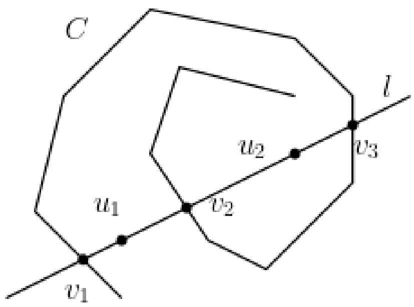


Figure 7.3: This illustrates the proof of Lemma 7.6.

Corollary 7.7 *Let P be a 2-convex polygon, and C be a reflex chain in P. Then vertices in C are in convex position.*

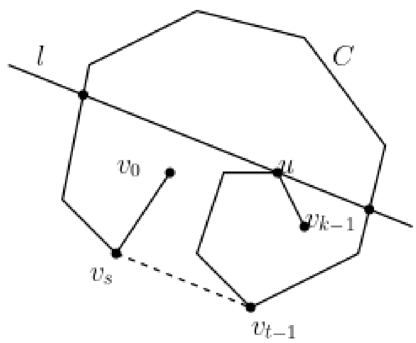


Figure 7.4: This figures illustrates the proof of Corollary 7.7.

Proof Let $C = (v_0, v_1, \cdots, v_{k-1})$, and $Q = conv(C)$. By Lemma 6.5, we further suppose that $V(Q) = \{v_s, v_{s+1}, \cdots, v_{t-1}\}, s < t$.

We prove this corollary by contradiction. Suppose the opposite that there exists vertex $u \in C$, such that $u \in int(Q)$. By the assumption, we can see that $C(v_s, v_{t-1}) \cup \overline{v_s v_{t-1}} = \partial Q$. Hence, let l be the line passing through u that is parallel to $\overline{v_s v_{t-1}}$. Since $u \in int(Q)$, we have that $cr(l, \partial Q) = 2$. Notice that $cr(l, \overline{v_s v_{t-1}}) = 0$, thereby we know $cr(l, C(v_s, v_{t-1})) = cr(l, \partial Q) = 0$. Hence, l also intersects with C at vertex u, which is a vertex of Q. Therefore, we have $cr(l, C) \geq cr(l, C(v_s, v_{t-1})) + 1 = 3$.

According to Lemma 7.6, as C is a reflex in the 2-convex polygon P, $cr(C, l)$ should be at most 2, which gives rise to the contradiction. □

Previous propositions are discussing about the arrangement of a reflex chain in a 2-convex chain, and we show that the vertices of such a reflex chain are always in convex position. However, when considering a convex chain, additional propositions must be established to achieve similar results.

Lemma 7.8 *Let P be a 2-convex polygon, and C be a convex chain in P. Then for any line l in the plane, we have $cr(C, l) \leq 4$.*

Proof This is implied by Lemma 10 in [6]. Any line l that intersects the convex chain C at least 5 times can be infinitesimally perturbed into a 6-stabber, which means that P is not 2-convex, leading to the contradiction. □

Corollary 7.9 *Let P be a 2-convex polygon, and $C = (v_0, v_1, \cdots, v_{k-1})$ be a convex chain of k vertices in P. Then there exists $X \subseteq V(C)$, such that $|X| \geq \lceil \frac{k}{3} \rceil$ and vertices in X are in convex position.*

Proof Let l be the line passing through vertex v_0 and v_{k-1}. By Lemma 7.8, we know that $2 \leq cr(C, l) \leq 4$. Accordingly, there are three different cases to consider.

- $cr(C, l) = 2$. This means that l only intersects with C at vertex v_0 and v_{k-1}. Thus, all the vertices in C are in convex position.
- $cr(C, l) = 3$. Suppose that other than v_0 and v_{k-1}, l intersects with C at point u, and $u \in \overline{v_t v_{t+1}}$. Further, let $C_1 = \{v_0, \cdots, v_t\}$ and $C_2 = \{v_{t+1}, \cdots, v_{k-1}\}$. We can see that vertices C_1 and C_2 are both in convex position, and $\max(|C_1|, |C_2|) \geq \frac{k}{2}$.
- $cr(C, l) = 4$. Suppose that other than v_0 and v_{k-1}, l intersects with C at point r and u, and $r \in \overline{v_s v_{s+1}}, u \in \overline{v_t v_{t+1}}$ and $s < t$. Let $C_1 = \{v_0, \cdots, v_s\}$, $C_2 = \{v_{s+1}, \cdots, v_t\}$ and $C_3 = \{v_{t+1}, \cdots, v_{k-1}\}$. Similarly, we can see that vertices in C_1, C_2, and C_3 are all in convex position, and $\max(|C_1|, |C_2|, |C_3|) \geq \frac{k}{3}$.

In summary, we can always find such set X including at least $\lceil \frac{k}{3} \rceil$ vertices, that are in convex position. □

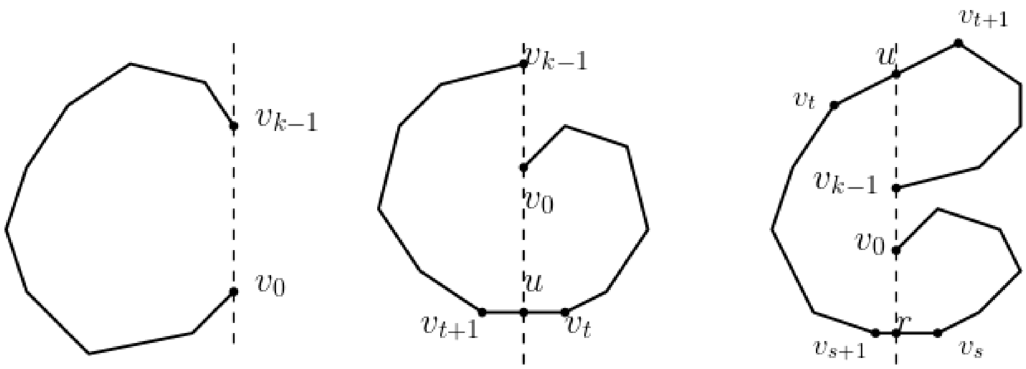


Figure 7.5: This figures shows the generic shape of three cases in the proof of Corollary 7.9.

Lemma 7.10 *Let P be a 2-convex polygon on n vertices, then there exists a subset of vertices $X \subseteq V(P)$, such that $|X| \geq \lceil \sqrt{n}/2 \rceil$ and vertices in X are in convex position*[1].

Proof Let k be the number of vertices in the convex hull of P, and let $Q = conv(P)$ be the convex hull of P.

By Lemma 7.4, between any two adjacent vertices in the convex hull, the reflex vertices are arranged consecutively. Thus, there exists at most k

[1]A similar result is reported in [6], but unfortunately the proof over there is flawed.

reflex chains in the boundary of P, such that all the reflex vertices of P are included in them. Meanwhile, any two consecutive reflex chains are also connected by a convex chain. Therefore, ∂P can be decomposed into at most k convex chains and k reflex chains.

If $k \geq \lceil \sqrt{n}/2 \rceil$, then let $X = V(Q)$, and X is already a set of vertices in convex position and $|X| \geq \lceil \sqrt{n}/2 \rceil$.

Otherwise, suppose that $k < \lceil \sqrt{n}/2 \rceil$, which means $k \leq \lfloor \sqrt{n}/2 \rfloor$. Notice that n edges in ∂P can be decomposed into at most k convex chains and k reflex chains. Among them, there exists either a convex chain of at least $3n/4k$ vertices, or a reflex chain of at least $n/4k$ vertices.

Suppose that there exists a convex chain C_1, and $|C_1| \geq \lceil \frac{3n}{4k} \rceil \geq \lceil \frac{3\sqrt{n}}{n} \rceil$. By Corollary 7.9, there exists $X \subseteq V(C_1)$, such that vertices of X are in convex position, and $|X| \geq \lceil |C_1|/3 \rceil \geq \lceil \sqrt{n}/2 \rceil$.

Similarly, suppose that there exists a reflex chain C_2, and $|C_2| \geq \lceil \frac{n}{4k} \rceil \geq \lceil \sqrt{n}/2 \rceil$. By Corollary 7.7, vertices in $X = V(C)$ are already in convex position.

Therefore, in each case mentioned above, we show that there exists $X \subseteq V(P)$, such that $|X| \geq \lceil \sqrt{n}/2 \rceil$ and vertices in X are in convex position. □

Remark 7.11 *This result is indeed extraordinary, as in a simple polygon of n vertices, only $\Omega(\log n)$ vertices are guaranteed to be in convex position. Further, this lower bound is asymptotically tight.*

7.3 Polygon with Two Hidden Points

In the previous section, we explored some fundamental properties of the 2-convex polygon. Now, we are prepared to step into the subclass of polygons that can accommodate only 2 hidden points. As indicated by Lemma 7.3, this class of polygons is a subset of the 2-convex polygon.

Proposition 7.12 *Let P be a simple polygon and $\mathrm{HP}(P) = 2$. Then all the reflex vertices in P are not adjacent to each other.*

Proof We prove this by contradiction, suppose that there exist vertices p_t and p_{t+1} in P such that both of them are reflex vertices. Then, we can see that $C = (p_{t-1}, p_t, p_{t+1}, p_{t+2})$ is indeed a reflex chain. Similar to the argument in Lemma 6.14, let q_i be the midpoint of $\overline{p_i p_{i+1}}$, and $\{q_{t-1}, q_t, q_{t+1}\}$ is indeed a hidden point set, contradicting to $\mathrm{HP}(P) = 2$. □

Proposition 7.13 *Let P be a simple polygon and* $\mathrm{HP}(P) = 2$*, and u, v be two distinct reflex vertices in P. Then, there exists vertex $r \in P(u, v)$ such that r is a vertex of the convex hull of P.*

Proof This is immediately implied by Lemma 7.4. Since $\mathrm{HP}(P) = 2$, any two reflex vertices in P should not be adjacent. This means that u and v belong to different intervals of reflex vertices. However, Lemma 7.4 show that, in the polygonal chain that connecting consecutive vertices in $conv(P)$, reflex vertices should be arranged consecutively. Therefore, there exists vertex $r \in P(u, v)$ such that $r \in V(conv(P))$. $\square$

Given the previous two propositions, we can show the generic shape of a simple polygon with $\mathrm{HP}(P) = 2$ as the following figure.

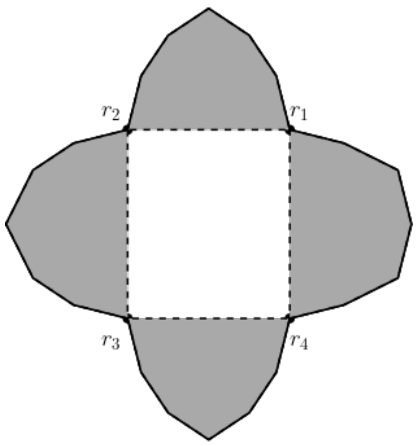


Figure 7.6: This figure shows the simple polygon P with $\mathrm{HP}(P) = 2$, which has in total 4 reflex vertices. The gray areas are the four convex pockets of P.

Let P be a simple polygon with $\mathrm{HP}(P) = 2$, and $R = \{v_1, v_2, \cdots, v_k\}$ be its reflex vertices in counter-clockwise order. Then, for any $i \in [k]$, one might think that vertices in $P(v_i, v_{i+1})$ are in convex position. Indeed, we can show a stronger result by proving the convex hull of this convex chain is part of the polygon P.

Definition 7.14 *Let P be a simple polygon with* $\mathrm{HP}(P) = 2$*, and $R = \{v_1, v_2, \cdots, v_k\}$ be its reflex vertices in counter-clockwise order. The i-th convex pocket of P, denoted as $\pi(P, i)$, is defined as the convex hull of vertices in $P(v_i, v_{i+1})$.*

To be specific, the *i*-th convex pocket is enclosed by the polygonal chain $P(v_i, v_{i+1})$ and the segment $\overline{v_i v_{i+1}}$.

Lemma 7.15 *Let* $P = (p_0, p_1, \cdots, p_{n-1})$ *be a simple polygon with* $\mathrm{HP}(P) = 2$*, and* $R = \{v_1, v_2, \cdots, v_k\}$ *be the set of its reflex vertices in counter-clockwise order. For any* $i \in [k]$*, we have* $\pi(P, i) \subseteq P$.

Proof We claim that for any i, vertices in $P(v_i, v_{i+1})$ are in convex position (the same fact was also claimed in [6]).

Then we suppose the contradiction that there exists $i \in k$ such that $\pi(P, i) \nsubseteq P$. Thus, there exists a vertex u of P other than $V(P(v_i, v_{i+1}))$ such that $u \in int(\pi(P, i))$.

Rewrite v_i as p_s, and v_{i+1} as p_t, $s < t$. Consider the line l passing through vertex w which is parallel to the segment $\overline{p_s p_t}$. Then we can see that l intersects with $P(p_s, p_t)$ at point u and r, where the four points p_s, u, r, and p_t are in counter-clockwise order. Without loss of generality, we could assume that the segment $\overline{ur}$ only intersects ∂P at points u, w, and r. Otherwise, we could translate l to pass another vertex $w' \in int(Q)$ which has more distance from $\overline{p_s p_t}$, or infinitesimally perturb the line l around the vertex w.

Denote the segment $\overline{ur}$ as z. Then, we can see that the segment z subdivides P into three parts P_1, P_2, and P_3. As is illustrated in the following figure, we have $\partial P_1 = P(r, w) \cup \overline{wr}, \partial P_2 = P(w, u) \cup \overline{uw}$, and $\partial P_3 = P(u, r) \cup \overline{ru}$.

Further, notice that for any point $p \in P_1 \setminus z, q \in P_2 \setminus z$, we have p and q are invisible to each other. Then, let $h_1 \in int(\overline{p_{s-1} p_s}), h_2 \in int(\overline{p_s p_{s+1}} \cap P(p_s, u)), h_3 \in int(\overline{p_{t-1} p_t} \cap P(p_t, r))$, and $h_4 \in int(\overline{p_t p_{t+1}})$ be four points in P. Then we can see that $\{h_1, h_2\} \subseteq P_2 \setminus z, \{h_3, h_4\} \subseteq P_1 \setminus z$. Therefore, $\{h_1, h_2\}$ are invisible to $\{h_3, h_4\}$.

Further, as both p_s and p_t are reflex vertices in P, we know that h_1 is invisible to h_2, and h_3 is invisible to h_4. Therefore, $H = \{h_1, h_2, h_3, h_4\}$ is a set of four hidden points, contradicting to $\mathrm{HP}(P) = 2$. □

Corollary 7.16 *Let* P *be simple polygon with* $\mathrm{HP}(P) = 2$*, and* $R = \{v_1, \cdots, v_k\}$ *be the set of its reflex vertices in the counter-clockwise order. Then, the polygon* $C = (v_1, v_2, \cdots, v_k)$ *is indeed a convex polygon, and* $C \subseteq P$.

Further, C *is defined as the kernel convex hull of* P*, denoted as* $\lambda(P)$.

Proof The proof shares almost the same idea with the one of Lemma 7.15, and it is omitted. □

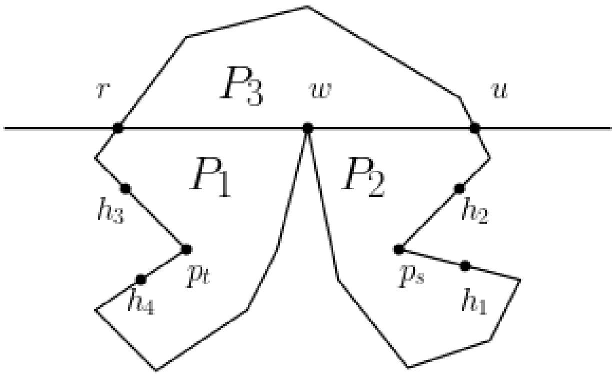


Figure 7.7: This illustrate the proof of Lemma 7.15.

Indeed, for a simple polygon P with $\mathrm{HP}(P) = 2$, which has k reflex vertices, it can be decomposed into $k+1$ convex pieces, without introducing any Steiner points. Specifically, $P = \lambda(P) \cup \pi(P,1) \cup \cdots \cup \pi(P,k)$.

Lemma 7.17 *Let P be a simple polygon with* $\mathrm{HP}(P) = 2$, *and* $R = \{v_1, \cdots, v_k\}$ *be the set of its reflex vertices in the counter-clockwise order. Then for any two non-adjacent convex pockets $\pi(P,i)$ and $\pi(P,j)$, which means v_i and v_j are not consecutive reflex vertices, we have vertices in $\pi(P,i)$ and $\pi(P,j)$ are in convex position.*

Proof For $i \in [k]$, rewrite v_i as p_{s_i}, and v_{i+1} as p_{t_i}. Further, for the i-th convex pocket, let $a_i = p_{s_i}, b_i = p_{s_i+1}, c_i = p_{t_i-1}$, and $d_i = p_{t_i}$. Then, we are about to show that both $L = (c_i, d_i, a_j, b_j)$ and $L' = (c_j, d_j, a_i, b_i)$ are convex chains, which is sufficient to prove that $C = (p_{s_i}, \cdots, p_{t_i}, p_{s_j}, \cdots, p_{t_j})$ is indeed a convex polygon.

As the cases of L and L' are symmetrical to each other, we only need to prove the case of L.

First of all, we need to be aware that $L = (c_i, d_i, a_j, b_j)$ is indeed a proper polygonal chain in P. By Corollary 7.16, we know that $\overline{d_i a_j} \subseteq \lambda(P) \subseteq P$. Hence, since $\overline{c_i d_i} \subseteq \partial P$ and $\overline{a_j b_j} \subseteq \partial P$, we have $L \subseteq P$.

Next, we prove L is a convex chain by contradiction. Suppose the opposite that L is not a convex chain. Then, L takes a right turn either at vertex d_i or at vertex a_j. Without loss of generality, we assume that L takes a right turn at vertex d_i, which means $C = (c_i, d_i, a_j)$ is a reflex chain in P. Therefore, c_i is invisible to a_j.

Further, denote $Vis(c_i)$ as Q, and thus $a_j \notin Q$. Then there exists $\varepsilon > 0$ such that $B(a_j, \varepsilon) \cap Q = \varnothing$. Denote such ball as C. Let $e_1 = \overline{c_{j-1} d_{j-1}}, e_2 =$

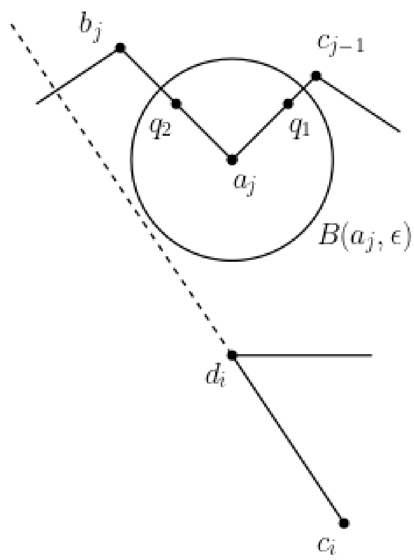


Figure 7.8: This figure illustrates the proof of Lemma 7.17.

$\overline{a_j b_j}$ be two edges that vertex a_j is incident to. Let q_1 and q_2 be two points such that $q_1 \in int(e_1 \cap C)$ and $q_2 \in int(e_1 \cap C)$. As a_j is a reflex vertex, (q_1, a_j, q_2) is a reflex chain and q_1 is invisible to q_2. Hence, since $q_1 \in C$ and $q_2 \in C$, we have that both q_1 and q_2 are invisible to c_i. Therefore, $\{c_i, q_1, q_2\}$ is a set of three hidden points, which contradicts to $\mathrm{HP}(P) = 2$. □

Corollary 7.18 *Let P be a simple polygon, and $\mathrm{HP}(P) = 2$. Let $\{m_1, \cdots, m_t\}$ be the set of non-adjacent convex pockets of P. Let $M = \bigcup_{i=1}^{t} V(m_i)$, then we have $conv(M) \subseteq P$.*

Proof For each m_i, let a_i and b_i be the reflex vertices included in m_i such that $\partial m_i = P(a_i, b_i) \cup \overline{a_i b_i}$. By Lemma 7.17, all the vertices in M are in convex position. Then, we can see that

$$conv(M) = (a_1, b_1, \cdots, a_t, b_t) \cup m_1 \cup \cdots \cup m_t.$$

Let $X = \bigcup_{i=1}^{t} \{a_i, b_i\}$, then X is a subset of reflex vertices in P. By Corollary 7.16, vertices in X are in convex position, and then $conv(X) = (a_1, b_1, \cdots, a_t, b_t) \subseteq \lambda(P) \subseteq P$.

Therefore, $conv(M) = conv(X) \cup m_1 \cup \cdots \cup m_t \subseteq P$. □

Eventually, we are ready to present the most important result in this section in the following lemma.

Lemma 7.19 *Let P be a simple polygon and* $\text{HP}(P) = 2$, *then* $\textsc{Cover}(P) \leq 3$.

Proof Let k be the number of reflex vertices, and $R = \{v_1, \cdots, v_k\}$ be the set of reflex vertices in counter-clockwise order. Let $m_i = \pi(P, i)$ be the i-th convex pocket of P. The cases for $k \leq 2$ is ordinary as $\textsc{Cover}(P) \leq k + 1 \leq 3$. The case for $k = 3$ is trivial as well. Thus, we always assume that $k \geq 4$. There are two different cases to consider.

- k is even, and let $k = 2t, t \geq 2$. Then, let $M_1 = \bigcup_{i=1}^{t} V(m_{2i-1}), Q_1 = conv(M_1)$, and $M_2 = \bigcup_{i=1}^{t} V(m_{2i}), Q_2 = conv(M_2)$.

  By Corollary 7.18, we know that $Q_1 \subseteq P$ and $Q_2 \subseteq P$. Further, for each convex pocket m_i, we have either $m_i \subseteq Q_1$ or $m_i \subseteq Q_2$. Hence, notice that $\lambda(P) \subseteq Q_1$, and $\lambda(P) \subseteq Q_2$ since all the vertices of $\lambda(P)$ are included in both M_1 and M_2. Thus, we have $P = Q_1 \cup Q_2$, implying that $\text{HP}(P) \leq 2$.

- k is odd, and let $k = 2t + 1, t \geq 2$. Similarly, let $M_1 = \bigcup_{i=1}^{t} V(m_{2i-1})$, $Q_1 = conv(M_1)$, and $M_2 = \bigcup_{i=1}^{t} V(m_{2i}), Q_2 = conv(M_2)$. Still, we have $Q_1 \subseteq P$ and $Q_2 \subseteq P$.

  Further, let $r = \overline{v_1 v_{2t}} \cap \overline{v_2 v_{2t+1}}$. Let $M_3 = \{r\} \cup V(m_{2t+1})$, and $Q_3 = conv(M_3)$. By Lemma 7.17, we have vertices in M_3 are in convex position.

  Denote the triangle (v_{2t+1}, v_1, r) as T, then we let $Q_3 = m_{2t+1} \cup T$. Thus, for each convex pocket m_i, we have $m_i \subseteq Q_1 \cup Q_2 \cup Q_3$. Consider the kernel convex hull of P, which is $\lambda(P) = (v_1, v_2, \cdots, v_{2t+1})$. Since vertices of $\lambda(P)$ are in convex position, we have

  $$\lambda(P) = \underbrace{(v_1, v_2, \cdots, v_{2t})}_{\subseteq Q_1} \cup \underbrace{(v_2, v_3, \cdots, v_{2t+1})}_{\subseteq Q_2} \cup \underbrace{(v_{2t+1}, v_1, r)}_{\subseteq Q_3}.$$

  Accordingly, we know $\lambda(P) \subseteq Q_1 \cup Q_2 \cup Q_3$, and hence $P = Q_1 \cup Q_2 \cup Q_3$, implying that $\text{HP}(P) \leq 3$.

In summary, given that $\text{HP}(P) = 2$, we always show that $\textsc{Cover}(P) \leq 3$, thus completing the proof. $\square$

But, this lemma only shows us the picture of a very special case. For cases where $\text{HP}(P) \geq 3$, which are considerably more complex, no definitive results have been established so far.

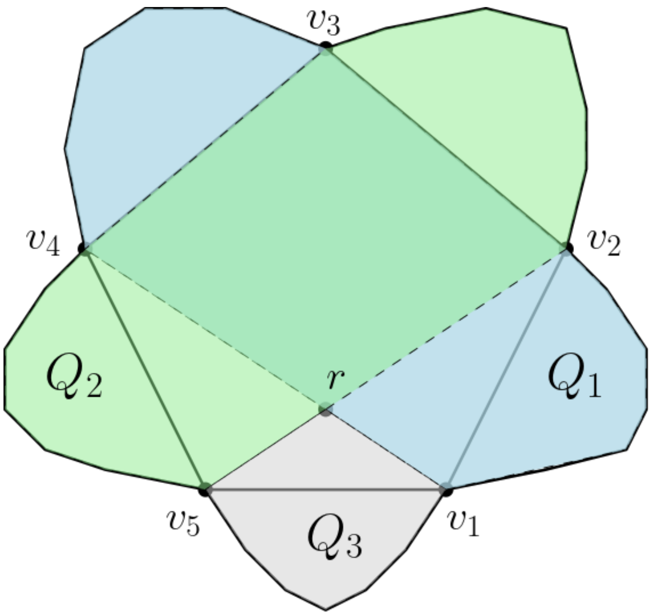


Figure 7.9: This figure shows the convex covering of P, which has 5 convex pockets. The blue, green, and grey areas are Q_1, Q_2, and Q_3 respectively, which follows the notation in the proof of Lemma 7.19.

Chapter 8

Funnel Polygon and PseudoTriangle

Previously, we took a close look at the spiral polygon, of which the boundary is composed of a convex chain and a reflex chain. Moreover, if we allow the polygon to have more convex and reflex chains, its structure would become more complicated. This exploration leads us to the funnel polygon and the pseudotriangle, which exhibit two and three reflex chains on their boundaries, respectively.

In this chapter, we will present brand new characterization on the visibility graph of them. Based on that, we will outline an efficient algorithm for finding the maximum hidden vertex set in the funnel polygon. Additionally, we will present a 2/3-approximation for the maximum hidden vertex set problem in the pseudotriangle. Note that prior to this, only a trivial 1/2-approximation was available.

8.1 Funnel Polygon

In this section, we are ready to discuss the funnel polygon[1]. The Funnel polygon is a fundamental geometric structure in lots of visibility algorithms, and also, it is a subclass of the 2-convex polygon.

Definition 8.1 *Let P be a simple polygon, P is called a funnel polygon if*

- *P only has three convex vertices,*
- *there exist convex vertices u and v in $V(P)$, such that $(u, v) \in E(P)$.*

[1]Someone calls it the tower polygon.

Such edge (u,v) is called a convex edge of P.

Let P be a funnel polygon. Without loss of generality, we always assume that P has only one convex edge. Otherwise, P will actually become a spiral polygon, since all the convex vertices of P are consecutive now, and of course we have already talked about spiral polygons before.

Accordingly, we note that the boundary of a funnel polygon can always be decomposed into two reflex chains and a convex edge.

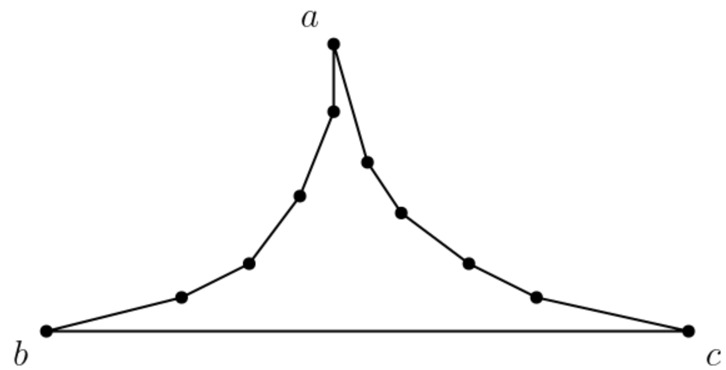


Figure 8.1: This figure show the shape of a funnel polygon P, where a,b,c are its convex vertices, and $(b,c) \in E(P)$.

Definition 8.2 *Let P be a funnel polygon, and a,b, and c be its convex vertices in counter-clockwise order such that $(b,c) \in E(P)$.*

The left reflex interval of P, denoted as $L(P)$, is defined as the set of vertices in $P(c,a)$. $L(P) := \{\lambda_0, \lambda_1, \cdots, \lambda_l\}$, and $(\lambda_l, \lambda_{l-1}, \cdots, \lambda_0) = P(c,a)$.

The right reflex interval of P, denoted as $R(P)$, is defined as the set of vertices in $P(a,b)$. $R(P) := \{\rho_0, \rho_1, \cdots, \rho_r\}$, and $(\rho_0, \rho_1, \cdots, \rho_r) = P(a,b)$.

Remark 8.3 *Different from our conventions before, vertices in $L(P)$ are actually indexed in clockwise order. Hence, we usually write $L(P)$ as L, and $R(P)$ as R, if the polygon P that we refer to is clear.*

Definition 8.4 *Let P be a funnel polygon, and L and R be its left and right reflex intervals respectively.*

For any $i \leq j$, the set of vertices between λ_i and λ_j, denoted as $L(i,j)$, is defined as $L(i,j) := \{\lambda_i, \lambda_{i+1}, \cdots, \lambda_j\} = V(P(\lambda_j, \lambda_i))$.

Similarly, for any $i \leq j$, $R(i,j) := \{\rho_i, \rho_{i+1}, \cdots, \rho_j\} = V(P(\rho_i, \rho_j))$.

8.1.1 Characterization of Funnel Polygon Visibility Graph

Following the definitions above, we begin to present new characterizations of the visibility graph of funnel polygon. At the very beginning, we show the following fact about reflex chains without a proof, and then establish our own arguments on the structure of funnel polygons.

Lemma 8.5 (Continuity of Visibility). *Let P be a simple polygon, and $L = (v_0, v_1, \cdots, v_{k-1})$ be a reflex chain in P. Let u be a point in P such that $I_P(u, v_0) = I_P(u, v_{k-1}) = 1$. Then, for any point $r \in L$, we have $I_P(u, r) = 1$ as well.*

Accordingly, given a point u and a reflex chain L, $Vis(u) \cap L$ is indeed a subchain of L. In other words, points in L that is visible to u is actually continuous, which make up a connected simple curve. By this continuity of the visibility, we immediately have the following propositions.

Proposition 8.6 *Let P be a funnel polygon, and L and R be the left and right reflex intervals respectively.*

Then, $\forall u \in L$, there exist $i \leq j$ such that $N(u) \cap R = R(i, j)$. Meanwhile, $\forall v \in R$, there exist $i \leq j$ such that $N(v) \cap L = L(i, j)$.

Proof Since R is a reflex chain in P, by Lemma 8.5, for any vertex $u \in L$, vertices of R that is visible to u should be consecutive. Thus, there exist indices $i \leq j$ such that $N(u) \cap R = R(i, j)$, and vice versa. □

Generally speaking, consider any vertex u, the vertices on the other reflex chain that are visible to u make up an interval. Further, we will prove the endpoints of such intervals is monotonely increasing through the following arguments.

Proposition 8.7 *Let $P = (p_0, \cdots, p_{n-1})$ be a funnel polygon, and (p_0, p_{n-1}) be the convex edge of P. Then, the polygonal chain $T = (p_0, p_1, \cdots, p_{n-1})$ is indeed a terrain.*

Proof We sketch the main idea. Let p_t be the convex vertex of P which is not incident to the convex edge. Further, we specify the y-axis by letting it parallel to $\overrightarrow{p_t p_{t+1}}$. After that, we specify the x-axis by letting it perpendicular to the y-axis.

One can see that, by this specification, T is indeed x-monotone. □

According to Proposition 8.7, we denote $P(p_0, p_{n-1})$ as T, and we $T = (\lambda_l, \cdots, \lambda_1, \lambda_0(\rho_0), \rho_1, \cdots, \rho_r)$. Further, we always assume that T is x-monotone, and the segment $\overline{\lambda_l \rho_r}$ lies above T.

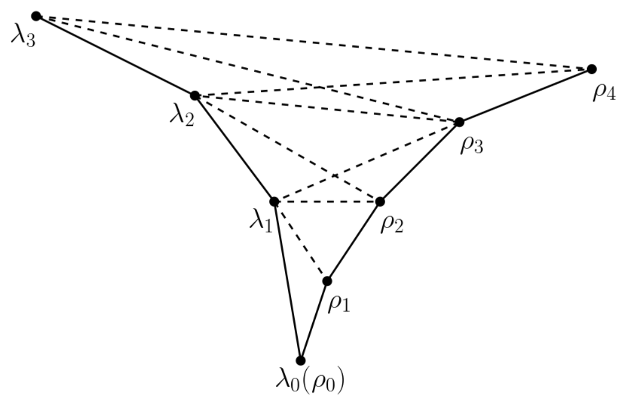


Figure 8.2: This figure illustrates the proof of Lemma 8.8, in which the bold polygonal chain represents the terrain. Meanwhile, the vertices are connected if and only if they are visible to each other.

Lemma 8.8 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a funnel polygon, such that*

- *the polygonal chain $T = (p_0, p_1, \cdots, p_{n-1})$ is a terrain,*
- *(p_0, p_{n-1}) is a convex edge of P, and $\overline{p_0 p_{n-1}}$ lies above T.*

Then, we have $VG(P) = VG(T)$.

Proof Notice that $\overline{p_0 p_{n-1}}$ lies above T, and p_0, p_{n-1} are the endpoints of T, then for any point u,

$$u \in P \iff u \text{ lies below } \overline{p_0 p_{n-1}} \wedge u \text{ lies above } T.$$

Accordingly, for any vertices u and v,

$$\overline{uv} \subseteq P \iff \overline{uv} \text{ lies below } \overline{p_0 p_{n-1}} \wedge \overline{uv} \text{ lies above } T.$$

Hence, notice that $\overline{uv}$ always lies below $\overline{p_0 p_{n-1}}$, we have

$$\overline{uv} \subseteq P \iff \overline{uv} \text{ lies above } T,$$
$$(u, v) \in VG(P) \iff (u, v) \in VG(T).$$

Thus we seal the conclusion that $VG(P) = VG(T)$. □

This lemma shows that the visibility graph of a funnel polygon is indeed the visibility graph of a terrain. Thus, we can utilize the known facts about terrain, to better our understanding about funnel polygons.

Corollary 8.9 *Let P be a funnel polygon, and $L = \{\lambda_i | i \in [0, l]\}$, $R = \{\rho_i | i \in [0, r]\}$ be its left and right reflex intervals.*

For any $0 \le a < b \le l, 0 \le c < d \le r$, if $I_P(\lambda_a, \rho_d) = 1$, and $I_P(\lambda_b, \rho_c) = 1$, then we have $I_P(\lambda_b, \rho_d) = 1$ as well.

Proof It is implied by the X-property of the terrain visibility graph [2].□

Proposition 8.10 *Let P be a funnel polygon, and $L = \{\lambda_i | i \in [0, l]\}$, $R = \{\rho_i | i \in [0, r]\}$ be its left and right reflex intervals.*

For any $i \in [0, l], j \in [0, r]$, if $I_P(\lambda_i, \rho_j) = 0$, then exact one of the following statement holds:

- $N(\lambda_i) \cap R(j, r) = \varnothing$,
- $N(\rho_j) \cap L(i, l) = \varnothing$.

Proof We sketch the main idea of the proof.

Denote the geodesic path from vertex λ_i to vertex ρ_j as C, then C is a reflex chain in P, which is implied by Theorem 1 in [28]. Let $X = V(C) \setminus \{\lambda_i, \rho_j\}$ be the intermediate vertices of C, then C takes a right turn at each vertex of X. Thus, we have either $X \subseteq L$, or $X \subseteq R$. Hence, note that $L \cap R = \{\lambda_0\}$, which only includes a convex vertex. Thus, $X \subseteq L$ and $X \subseteq R$ are disjoint events, and exact one of them happens.

If $X \subseteq L$, then we have $N(\lambda_i) \cap R(0, j) = \varnothing$, and $N(\rho_j) \cap L(i, l) = \varnothing$.

Otherwise, if $X \subseteq R$, then we have $N(\lambda_i) \cap R(j, r) = \varnothing$, and $N(\rho_j) \cap L(0, i) = \varnothing$, which completes the proof. □

Finally, we can derive the monotonicity for the visibility intervals in a funnel polygon.

Lemma 8.11 *Let P be a funnel polygon, and $L = \{\lambda_i | i \in [0, l]\}$, $R = \{\rho_i | i \in [0, r]\}$ be its left and right reflex intervals.*

For all $i \in [0, l]$, let α_i, β_i be the integers such that $N(\lambda_i) \cap R = R(\alpha_i, \beta_i)$.

For all $i \in [0, r]$, let γ_i, δ_i be the integers such that $N(\rho_i) \cap L = L(\gamma_i, \delta_i)$.

Then, the four sequences $\{\alpha_i\}_{i=0}^{l}, \{\beta_i\}_{i=0}^{l}, \{\gamma_i\}_{i=0}^{r}$, and $\{\delta_i\}_{i=0}^{r}$ are all monotonely increasing.

Proof It suffices to show that $\forall i \leq l - 1, \alpha_i \leq \alpha_{i+1}$, and $\beta_i \leq \beta_{i+1}$, since the argument for γ_i and δ_i is the same.

First we argue that $\forall i \leq l - 1, \beta_i \leq \beta_{i+1}$. For the sake of contradiction, suppose there exists k such that $\beta_k > \beta_{k+1}$. Let $t = \beta_k$, and $s = \beta_{k+1}$. By the definition, we know that $I_P(\lambda_k, \rho_t) = I_P(\lambda_{k+1}, \rho_s) = 1$. By Corollary 8.9, we know that $I_P(\lambda_{k+1}, \rho_t) = 1$, which means λ_{k+1} is also visible to ρ_t. However, since $\beta_{k+1} = s < t$, we know $\rho_t \notin N(\lambda_{k+1})$, which means λ_{k+1} is actually invisible to ρ_t, arriving at a contradiction.

Next we will show that $\forall i \leq l-1, \alpha_i \leq \alpha_{i+1}$. For the sake of contradiction, suppose there exists k such that $\alpha_k > \alpha_{k+1}$. Let $t = \alpha_k$, and $s = \alpha_{k+1}$. By the definition, we have $I_P(\lambda_k, \rho_t) = I_P(\lambda_{k+1}, \rho_s) = 1$ and $I_P(\lambda_k, \rho_s) = 0$. By the 8.10, we should have either $N(\lambda_k) \cap R(s,r) = \varnothing$, or $N(\rho_s) \cap L(k,l) = \varnothing$. However, we can already see that $\rho_t \in N(\lambda_k) \cap R(s,r) \neq \varnothing$, and $\lambda_{k+1} \in N(\rho_s) \cap L(k,l) \neq \varnothing$, thus leading to the contradiction. $\square$

The significance of the above arguments lies in the fact that the continuity and monotonicity of the visibility intervals in the funnel polygon provide us with a favorable opportunity to efficiently solve hard visibility problems. For instance, an $O(n^4)$ algorithm has been proposed for figuring out the minimum dominating set in a funnel polygon [25]. Further, it is worth noting that its time complexity can be improved to $O(n^2)$, because the $O(n^4)$ algorithm only observed the continuity, but put the monotonicity aside.

In addition to investigating the local structure of the visible area, we are also interested in having a global perspective on the visibility graph of funnel polygons. We can explore this aspect further through the following propositions.

Definition 8.12 *Let G be a simple graph, G is called a weakly triangulated graph if neither G nor $\overline{G}$ contains a chordless circle with length at least five.*

Theorem 8.13 *Let G be a weakly triangulated graph, then G is perfect.*

Proof This is proved in [30]. $\square$

Lemma 8.14 *Let P be a funnel polygon, then $VG(P)$ is a weakly triangulated graph, and thus a perfect graph.*

Proof This is proved in [14]. $\square$

Lemma 8.15 *Let P be a funnel polygon, and G be its visibility graph. Then G is K_5-free.*

Proof Let $G = VG(P)$ be its visibility graph. Suppose the opposite that there exists a clique C in G, and $|C| \geq 5$.

Let L and R be the left and right intervals of P, then $V(P) = L \cup R$, $C = (C \cap L) \cup (C \cap R)$. Thus, $\max\{|C \cap L|, |C \cap R|\} \geq 3$, and suppose that $|C \cap L| \geq 3$. By this, we can see that $X = C \cap L$ is a clique of size at least three in L. However, since the vertices of L compose a reflex chain in P, we know that $G[L]$ is actually triangle-free, thus leading to the contradiction. $\square$

Corollary 8.16 *Let P be a funnel polygon on n vertices, then* $\mathrm{HV}(P) \geq \frac{n}{4}$.

Proof Let G be the visibility graph of P, then by Lemma 8.14, G is weakly triangulated and perfect. Accordingly, $\mathrm{HV}(P) = \alpha(G) = \kappa(G)$.

By Lemma 8.15, we have G is K_5-free, and thus $\omega(G) \leq 4$. Therefore, we have $\kappa(G) \geq n/\omega(G) \geq \frac{n}{4}$, implying that $\mathrm{HV}(P) \geq \frac{n}{4}$. □

8.1.2 Hidden Points and Hidden Vertices in Funnel Polygon

In this subsection, we will talk about how to figure out the maximum hidden vertex set and the maximum hidden point set in a funnel polygon.

Lemma 8.17 *Let P be a funnel polygon, and $L = \{\lambda_i | i \in [0, l]\}$, $R = \{\rho_i | i \in [0, r]\}$ be its left and right reflex intervals.*

For any $i \in [0, l], j \in [0, r]$, let $f(i, j)$ denote the size of maximum hidden vertex set in $L(0, i) \cup R(0, j)$.

Then, we can compute $\{f(i, j) | \forall i \in [0, l], j \in [0, r]\}$ in time complexity $O(n^2)$.

Proof We first present our algorithm, then prove its correctness and efficiency.

For each $i \in [0, l]$, let α_i be the smallest integer such that $I_P(\lambda_i, \rho_{\alpha_i}) = 1$.

For each $j \in [0, r]$, let γ_j be the smallest integer such that $I_P(\lambda_{\gamma_j}, \rho_j) = 1$.

If either $i = 0$ or $j = 0$, $f(i, j)$ will be trivial be compute as $L(0, i) \cup R(0, j)$ is the set of vertices in a reflex chain.

Otherwise, we suppose that $i \geq 1$ and $j \geq 1$. Then, we consider following different cases.

- $I_P(\lambda_i, \rho_j) = 1$, then let $f(i, j) = \max\{f(i-1, j), f(i-2, \alpha_i - 1) + 1\}$.
- $I_P(\lambda_i, \rho_j) = 0$. By Proposition 8.10, we have either $\alpha_i > j$ or $\gamma_j > i$. If $\alpha_i > j$, then let $f(i, j) = \max\{f(i-1, j), f(i-2, j) + 1\}$. Otherwise, if $\gamma_j > i$, let $f(i, j) = \max\{f(i, j-1), f(i, j-2) + 1\}$.

In summary, we have

$$f(i, j) = \begin{cases} \max\{f(i-1, j), f(i-2, \alpha_i - 1) + 1\}, & I_P(\lambda_i, \rho_j) = 1, \\ \max\{f(i-1, j), f(i-2, j) + 1\}, & \alpha_i > j, \\ \max\{f(i, j-1), f(i, j-2) + 1\}, & \gamma_j > i. \end{cases}$$

To prove the correctness, we need to argue that $f(i,j) = \alpha(G[L(0,i) \cup R(0,j)])$. We denote such proposition as $\varphi(i,j)$, and we prove it by induction on $i+j$.

For $0 \leq i+j \leq 2$, we assume that $\varphi(i,j)$ holds. Since there are only constant number of vertices to consider, such $f(i,j)$ can be computed in $O(1)$ time. Further, for either $i=0$ or $j=0$, we assume $\varphi(i,j)$ always holds as well, since in such case $f(i,j)$ is also trivial to compute.

Suppose that $\forall i+j \leq k$, $\varphi(i,j)$ holds. Then let's consider the case for $i+j = k+1$.

Let X be the maximum hidden vertex set of $L(0,i) \cup R(0,j)$, and we argue that $f(i,j) = |X|$. There are following different cases to consider.

- $I_P(\lambda_i, \rho_j) = 1$, then we have $\alpha_i \leq j$. In this case, if $\lambda_i \in X$, then $X \setminus \{\lambda_i\}$ is indeed maximum hidden vertex set in $L(0, i-2) \cup R(0, \alpha_i - 1)$, thus by induction we have $|X| = f(i-2, \alpha_i - 1)$. Otherwise, if $\lambda_i \notin X$, then X is the maximum hidden vertex set in $L(0, i-1) \cup R(0,j)$, and $|X| = f(i-1,j)$. Therefore, $f(i,j) = |X| = max\{f(i-1,j), f(i-2, \alpha_i - 1) + 1\}$, $\varphi(i,j)$ still holds.
- $I_P(\lambda_i, \rho_j) = 0$, then by Proposition 8.10, we have either $\alpha_i > j$, or $\gamma_j > i$, and exact one of them is true. We only prove the case for $\alpha_i > j$, as the argument for γ_j is almost the same. Suppose that $\alpha_i > j \geq 1$, and this implies $i \geq 2$, since otherwise we should have $\alpha_i = 0$. In this case, if $\lambda_i \notin X$, then X is the maximum hidden vertex set in $L(0, i-1) \cup R(0,j)$, and $|X| = f(i-1,j)$. Otherwise, we have $\lambda_i \in X$. Notice that the only vertex in $L(0, i-1) \cup R(0,j)$ that is visible to λ_i is λ_{i-1}. Thus, $X \setminus \{\lambda_i\}$ is the maximum hidden vertex set in $L(0, i-2) \cap R(0,j)$, and $f(i,j) = |X| = max\{f(i-1,j), f(i-2,j) + 1\}$, implying that $\varphi(i,j)$ still holds for this case.

Therefore, we can see that in both cases $\varphi(i,j)$ remains true when $i+j = k+1$, thus completing the induction. □

Indeed, by Lemma 8.8, we can see that the visibility graph of a funnel polygon is also the visibility graph of a terrain. Therefore, finding the maximum hidden vertex in the funnel polygon is the same as in the terrain. An alternative solution for finding the maximum hidden vertex set in a terrain will be illustrated in the next chapter, which merely relies on the persistence of the terrain.

When it comes to the maximum hidden point set, similar arguments can be established by the following proposition.

Lemma 8.18 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a funnel polygon, and*

- *the polygonal chain $T = (p_0, p_1, \cdots, p_{n-1})$ is a terrain,*
- *(p_0, p_{n-1}) is a convex edge of P.*

Let H be a hidden point set in P, then there always exists another hidden point set H', such that $|H'| = |H|$ and $H' \subseteq T$.

Proof Without loss of generality, suppose that $\overline{p_0 p_{n-1}}$ lies above T.

Let $H = \{h_1, \cdots, h_k\}$, and $h_i = (x_i, y_i)$. For each $i \in [k]$, let $h'_i = (x'_i, y'_i)$ be the point in T such that $x_i = x'_i$. Note that such h'_i always exists and is unique in T. Further, since $h_i \in P$, we have h_i lies above T, and thus $y'_i \leq y_i$.

For any $i \neq j$, we have $x_i \neq x_j$, otherwise h_i is visible to h_j. Hence, since $\overline{h_i h_j} \not\subseteq P$ and $\overline{h_i h_j}$ lies below $\overline{p_0 p_{n-1}}$, we know that $\overline{h_i h_j}$ does not completely lie above T. Thus, there exists vertex r, such that r lies strictly above $\overline{h_i h_j}$. Meanwhile, since $y'_i \leq y_i$ and $y'_j \leq y_j$, we have r also lies strictly above $\overline{h'_i h'_j}$, thus making $\overline{h'_i h'_j} \not\subseteq P$.

Therefore, $H' = \{h'_1, \cdots, h'_k\}$ is indeed a hidden point set in P, and by definition, $|H'| = |H|$ and $H' \subseteq T$. ☐

In fact, such H' is also a hidden point set with regard to the terrain T. Accordingly, maximum hidden point set in a funnel polygon, again, shares the same size with maximum hidden point set in its corresponding terrain. In the next chapter, we will illustrate how to solve it efficiently in polynomial time.

In fact, recently, a linear time algorithm has been proposed in [9] to figure out both HP(P) and HV(P) for a funnel polygon spontaneously, which is purely based on explicit geometric constructions. The following proposition is immediately implied by it.

Lemma 8.19 *Let P be a funnel polygon on n vertices, then* HP(P) *and* HV(P) *is computable in $O(n)$ time.*

But still, we can get beneficial implications from our algorithm, especially for the scenario in the pseudotriangle, which is a natural generalization of the funnel polygon.

8.2 Pseudotriangle

In this section, we focus on the maximum hidden vertex in the pseudotriangle. We will propose an approximation algorithm in polynomial time, which gives us a $\frac{2}{3}$-competitive solution. Prior to this, the best known competitive ratio is $\frac{1}{2}$ in [9]. Our algorithm notably improves it, and provides the first non-trivial competitive ratio so far.

Definition 8.20 *Let P be a simple polygon, P is called a pseudotriangle if P only has three convex vertices.*

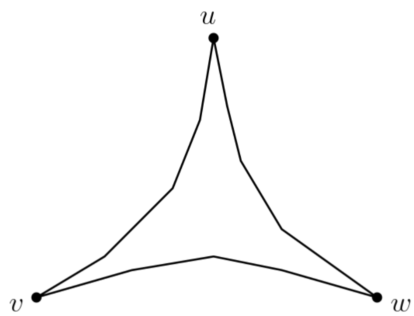


Figure 8.3: This illustrates the typical shape of a pseudotriangle P, where u, v, w are the reflex vertices of it.

Let P be pseudotriangle, and u, v, w be its convex vertices in counter-clockwise order. We always assume that P does not have any convex edge. Thus, we can see that ∂P can be decomposed into three reflex chains, subdivided by the reflex vertices u, v, and w.

By definition, the pseudotriangle is a superclass of the funnel polygon. In fact, a pseudotriangle can be constructed by substituting the convex edge with a reflex chain in a funnel polygon. Through this similarity of the structure, similar propositions can be argued for the pseudotriangle.

Definition 8.21 *Let P be a pseudotriangle, u be a convex vertex in it. Denote the other two convex vertices as u, w, and u, v, w are in counter-clockwise order.*

The left reflex interval with regard to vertex u, denoted as L_u, is defined as the set of vertices in $P(u, v)$. $L_u = \{\lambda_0, \lambda_1, \cdots, \lambda_l\}$, and $(\lambda_0, \lambda_1, \cdots, \lambda_l) = P(u, v)$.

The right reflex interval with regard to vertex u, denoted as R_u, is defined as the set of vertices in $P(w, u)$. $R_u = \{\rho_0, \rho_1, \cdots, \rho_r\}$, and $(\rho_r, \rho_{r-1}, \cdots, \rho_0) = P(w, u)$.

Then, $L_u(i, j) := \{\lambda_i, \lambda_{i+1}, \cdots, \lambda_j\}$, and $R_u(i, j) := \{\rho_i, \rho_{i+1}, \cdots, \rho_j\}$.

Lemma 8.22 *Let P be a pseudotriangle, and u be a convex vertex of P. Let L_u, R_u, $\{\lambda_i\}_{i=0}^{l}$, $\{\rho_i\}_{i=0}^{r}$ follow the definitions and notations in Definition 8.21.*

Then we have the following propositions:

- $\forall i \in [0, l]$, *there exists α_i, β_i, such that* $N(\lambda_i) \cap R_u = R_u(\alpha_i, \beta_i)$.
- $\forall j \in [0, r]$, *there exists γ_j, δ_j, such that* $N(\rho_j) \cap L_u = L_u(\gamma_j, \delta_j)$.
- *The sequences $\{\alpha_i\}_{i=0}^{l}$ and $\{\gamma_j\}_{j=0}^{r}$ are both monotonely increasing.*

The proof itself mostly agrees with our arguments for the funnel polygon, so we do not present it here again.

Compared to the funnel polygon, $\{\alpha_i\}$ and $\{\gamma_i\}$ remains monotonely increasing, while $\{\beta_i\}$ and $\{\delta_i\}$ might no longer be monotonely increasing. The reason is that $\{\alpha_i\}$ and $\{\gamma_i\}$ only depends the local structure of L_u and R_u, which is actually the same as the funnel polygon. However, $\{\beta_i\}$ and $\{\delta_i\}$ are jointly determined by all reflex chains, as the window in its visible area can also be incident to the reflex vertex on the third chain. See the following figure for illustration.

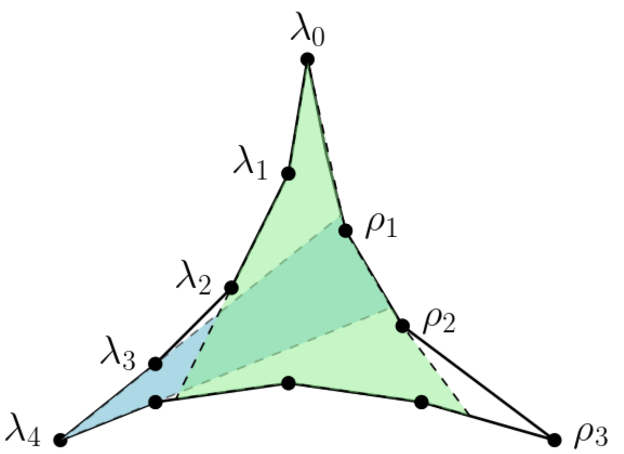


Figure 8.4: This figures illustrate why $\{\beta_i\}_{i=0}^{l}$ might no longer be monotonely increasing. The green and blue area are the visible area of λ_1 and λ_4, where $\beta_2 = 2$, $\beta_4 = 1$, and $\beta_2 > \beta_4$

.

Take $\{\beta_i\}$ as an example. Indeed, the increase and decrease in β_i provide us with much information about the geometric structure. Let k be a integer, and $s = \beta_k, t = \beta_{k+1}$. If $s < t$, then we know that λ_k can not see ρ_{s+1} because its vision is blocked by the vertex ρ_s, which means ρ_s is incident to a window in $Vis(\lambda_k)$. However, if $s > t$, then we can say that

λ_{k+1} can not see ρ_{t+1} because its vision gets blocked by a vertex r on the third reflex chain. See the following figures for detailed illustration.

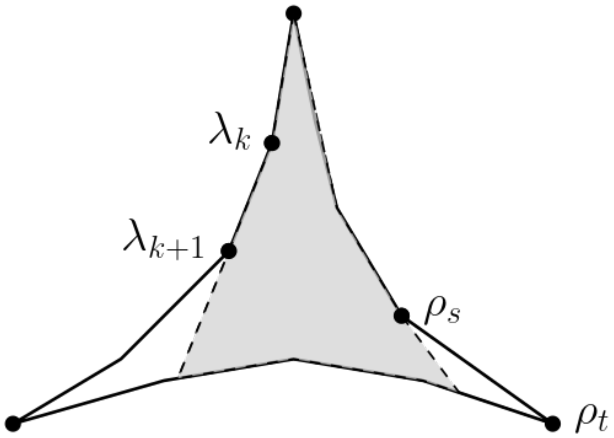


Figure 8.5: This illustrates the case when $\beta_k < \beta_{k+1}$. The grey area indicates the visible area of λ_k, where $Vis(\lambda_k)$ has a window incident to the reflex vertex ρ_s.

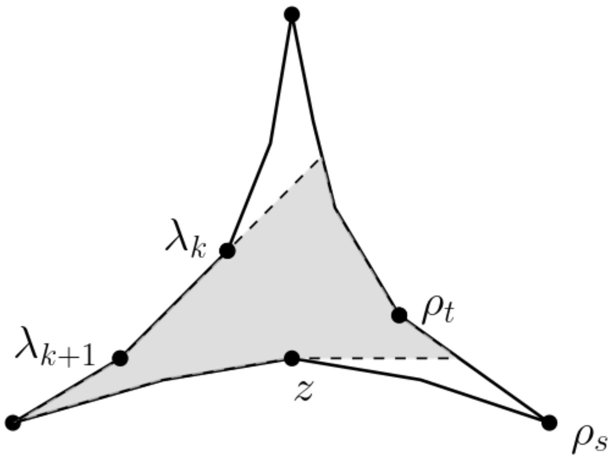


Figure 8.6: This illustrates the case when $\beta_k > \beta_{k+1}$. The grey area indicates the visibility area of λ_{k+1}. One can see that $Vis(\lambda_{k+1})$ has a window incident to reflex vertex z, while $z \notin L_u$, and $z \notin R_u$.

In fact, the monotonicity in $\{\beta_i\}$ and $\{\gamma_i\}$ indicates the structure of the visible area of λ_i and ρ_i, and itself can be characterized by the following proposition.

Proposition 8.23 *Let P be a pseudotriangle, and u be a convex vertex of P. Let $\{\beta_i\}_{i=0}^{l}$ and $\{\delta_i\}_{i=0}^{r}$ follow the definition and notation in Lemma 8.22. Then we have the following propositions:*

- *$\exists\, k \in [0, l]$, such that $\forall i < k, \beta_i \leq \beta_{i+1}$ and $\forall i \geq k, \beta_i \geq \beta_{i+1}$.*
- *$\exists\, k \in [0, r]$, such that $\forall i < k, \gamma_i \leq \gamma_{i+1}$ and $\forall i \geq k, \gamma_i \geq \gamma_{i+1}$.*

8.2.1 (2/3)-Approximation for Maximum Hidden Vertex Set in PseudoTriangle

Through the above propositions, we can compute the maximum hidden vertex set for $L_u \cup R_u$ via dynamic programming.

Definition 8.24 *Let X be a set of finite number of integers, X is a proper set if one of the following stands:*

- *There exists $i \geq 0$ such that $X = [0, i]$.*
- *There exists $i, j, k \geq 0$, $i + 2 \leq j \leq k$, such that that $X = [0, i] \cup [j, k]$.*

In other words, X is consider to be a proper set if it is composed by two groups of consecutive integers, with one of them including the zero.

Definition 8.25 *Let X be a proper set, and we define*

$$\begin{aligned}
&\tau(X) := \max\{x | x \in X\}, \\
&\sigma(X) := [0, \tau(X)], \\
&\eta(X) := \sigma(X) \setminus X.
\end{aligned}$$

Accordingly, if $X = [0, i] \cup [j, k], i + 2 \leq j$, then we have $\sigma(X) = [0, k]$ and $\eta(X) = [i + 1, j - 1]$.

Definition 8.26 *Let P be a pseudotriangle, and u be a convex vertex of P. Let $L_u = \{\lambda_i\}_{i=0}^{l}$, $R_u = \{\rho_i\}_{i=0}^{r}$ be the left and right reflex intervals with regard to vertex u.*

Let X be a set of integers and $X \subseteq [0, l]$. Then, the subset of L_u induced by X, denoted by $L_u[X]$, is defined as $L_u[X] := \{\lambda_i | i \in X\}$.

Let Y be a set of integers and $Y \subseteq [0, r]$. Then, the subset of R_u induced by Y, denoted by $R_u[Y]$, is defined as $R_u[Y] := \{\rho_i | i \in Y\}$.

Definition 8.27 *Let P be a pseudotriangle, and u be a convex vertex of P. Let $L_u = \{\lambda_i\}_{i=0}^{l}$, $R_u = \{\rho_i\}_{i=0}^{r}$ be the left and right reflex intervals with regard to vertex u.*

Let X, Y be two sets of integers, then (X, Y) is called a proper pair with regard to vertex u if and only if

- *$X \subseteq [0, l]$, and X is a proper set.*
- *$Y \subseteq [0, r]$, and Y is a proper set.*
- *$\exists\ S \subseteq [\tau(X) + 2, l]$, such that $N(L_u[S]) \cap R_u[\sigma(Y)] = R_u[\eta(Y)]$.*
- *$\exists\ T \subseteq [\tau(Y) + 2, r]$, such that $N(R_u[T]) \cap L_u[\sigma(X)] = L_u[\eta(X)]$.*

After introducing the preliminary definitions and notations, we are eventually ready to present our algorithm.

Lemma 8.28 *Let P be a pseudotriangle on n vertices, and u be a convex vertex of P. Let $L_u = \{\lambda_i\}_{i=0}^{l}$, $R_u = \{\rho_i\}_{i=0}^{r}$ be the left and right reflex intervals with regard to vertex u.*

Denote the set of all proper pairs with regard to u as M. For any $(X, Y) \in M$, denote $f(X, Y)$ be the size of maximum hidden vertex set in $L_u[X] \cup R_u[Y]$.

Then, $\{f(X, Y) | (X, Y) \in M\}$ can be computed in time $O(n^6)$.

Proof We first present our algorithm, and then prove its correctness and efficiency.

Let $\{\alpha_i\}_{i=0}^{l}$, $\{\beta_i\}_{i=0}^{l}$, $\{\gamma_i\}_{i=0}^{r}$, and $\{\delta_i\}_{i=0}^{r}$ follow the definitions and notations in Definition 8.21.

For the case either $|X| \leq 2$ or $|Y| \leq 2$, $f(X, Y)$ is trivial to compute, as the optimal solution in this case will be a set of vertices on the same reflex chain.

Thus, we always assume that $|X| > 2$ and $|Y| > 2$. Denote $\tau(X)$ as i, and $\tau(Y)$ as j. Then there are following different scenarios to consider.

$\mathcal{A}$. $I_P(\lambda_i, \rho_j) = 1$. Then $j \in [\alpha_i, \beta_i]$, thus we have $Y \setminus [\alpha_i, \beta_i]$ is indeed proper set. Thus, let

$$f(X, Y) = \max\{f(X \cap [0, i-1], Y), f(X \cap [0, i-2], Y \setminus [\alpha_i, \beta_i]) + 1\}.$$

$\mathcal{B}$. $I_P(\lambda_i, \rho_j)$, indicating that λ_i is invisible to ρ_j, and this includes three subclass cases.

- (a) $[\alpha_i, \beta_i] \cap \sigma(Y) = \varnothing$. Then, no vertex in $\sigma(Y)$ is visible to λ_i. Thus, we let

  $$f(X, Y) = \max\{f(X \cap [0, i-1], Y), f(X \cap [0, i-2], Y) + 1\}.$$

- (b) $[\gamma_j, \delta_j] \cap \sigma(X) = \varnothing$. Similar to the case (a), we let

  $$f(X, Y) = \max\{f(X, Y \cap [0, j-1]), f(X, Y \cap [0, j-2]) + 1\}.$$

- (c) $[\alpha_i, \beta_i] \subseteq \sigma(Y)$ and $[\gamma_j, \delta_j] \subseteq \sigma(X)$.

  In this case, if $X = \sigma(X)$, then $X \setminus [\gamma_j, \delta_j]$ is another proper set, and we let

  $$f(X,Y) = \max\{f(X, Y \cap [0, j-1]), f(X \setminus [\gamma_j, \delta_j], Y \cap [0, j-2]) + 1\}.$$

  Otherwise, if $Y = \sigma(Y)$, then $Y \setminus [\alpha_i, \beta_i]$ is another proper set, and we let

  $$f(X,Y) = \max\{f(X \cap [0, i-1], Y), f(X \cap [0, i-2], Y \setminus [\alpha_i, \beta_i]) + 1\}.$$

  Otherwise, we have $X \neq \sigma(X)$ and $Y \neq \sigma(Y)$, imply that $\eta(X) \neq \varnothing$ and $\eta(Y) \neq \varnothing$.

  In this case, we will argue that both $\eta(X) \cap [\gamma_j, \delta_j] \neq \varnothing$ and $\eta(Y) \cap [\alpha_i, \beta_i] \neq \varnothing$, which means both $X \setminus [\gamma_j, \delta_j]$ and $Y \setminus [\alpha_i, \beta_i]$ are proper sets.

  Let's consider $\eta(Y) \cap [\alpha_i, \delta_i]$, and denote $\eta(Y) = [s, t]$. Since (X, Y) is a proper pair, by the definition, there exists $k \geq i + 2$, such that $\alpha_k = s$. Suppose that $\beta_i \geq s$, then we are already done since $\eta(Y) \cap [\alpha_i, \delta_i] \neq \varnothing$. Otherwise, we have $\beta_i < s$, and thus $\beta_i < \beta_k = s$. By Proposition 8.23, $\forall i' \leq i$, we have $\beta_{i'} \leq \beta_i < s$. Thus, any vertex in $L_u[\sigma(X)]$ is invisible to ρ_j. Therefore, we have $[\gamma_j, \delta_j] \cap \sigma(X) = \varnothing$, and this situation belongs to the case (b).

  Similarly, if $\eta(X) \cap [\gamma_j, \delta_j] = \varnothing$, then we have $[\alpha_i, \beta_i] \cap \sigma(Y) = \varnothing$, and actually this belongs to (a).

  Thus, $Y \setminus [\alpha_i, \beta_i]$ is a proper set, and we let

  $$f(X,Y) = \max\{f(X \cap [0, i-1], Y), f(X \cap [0, i-2], Y \setminus [\alpha_i, \beta_i]) + 1\}.$$

By the above discussion, we conclude every possible situation via dynamic programming.

To prove the correctness of it, we just need to apply induction on the $|X| + |Y|$, and consider every possible situation of the optimal solution. It has almost the same idea and procedure as the one of Lemma 8.17, so we do not present it here.

Finally, we are about to analyze the time complexity of it. Notice that both L_u and R_u have less than n vertices, there are at most n^3 proper sets in $[0, l]$ and $[0, r]$, and in total at most n^6 proper pairs with regard to vertex u.

Further, for each proper pair (X, Y), it takes $O(1)$ time to compute it via the dynamic programming. Thus, the total complexity is $O(n^6)$, thus completing the proof. $\square$

Theorem 8.29 *Let P be a pseudotriangle with n vertices, then a hidden vertex set X, with $|X| \geq \frac{2}{3}\mathrm{HV}(P)$, can be computed in $O(n^6)$.*

Proof Let v_1, v_2, v_3 be the convex vertices of P. For each i, let L_i and R_i be left and right reflex intervals with regard to v_i, and $S_i = L_i \cup R_i$.

Let OPT be the maximum hidden vertex set of P, then we have

$$\sum_{i=1}^{3} |OPT \cap S_i| \geq 2|OPT|$$

since each vertex appears at least twice in all the S_i.

Further, let f_i be the size of maximum hidden vertex set in S_i, which by Lemma 8.28 can be computed in $O(n^6)$ time. Hence, since $OPT \cap S_i$ is also a hidden vertex set in S_i, we have

$$\sum_{i=1}^{3} f_i \geq \sum_{i=1}^{3} |OPT \cap S_i| \geq 2|OPT|.$$

Therefore,

$$\max\{f_1, f_2, f_3\} \geq \frac{2}{3}|OPT|,$$

which grants us a hidden vertex set X such that $|X| \geq \frac{2}{3}\mathrm{HV}(P)$. $\square$

Theorem 8.30 *Let P be a simple polygon with n vertices, including c convex vertices, then a hidden vertex set X, with $c|X| \geq 2\mathrm{HV}(P)$, can be computed in $O(n^6)$.*

Proof This theorem is indeed an immediate generalization of Theorem 8.29, and we sketch the main idea of it.

Let $\{v_i\}_{i=1}^{c}$ be the convex vertices of P in counter-clockwise order. Let L_i and R_i be left and right reflex intervals with regard to v_i, and $S_i = L_i \cup R_i$. Further, let t_i denote the number of vertices in $P(v_i, v_{i+1})$.

For each $i \in [k]$, let l_i be the geodesic path from v_{i+1} to v_{i-1}. Then, l_i is indeed a reflex chain, without any intermediate vertices in S_i. Thus, the closed polygonal chain $l_i \cup P(v_{i-1}, v_{i+1})$ indeed encloses a pseudotriangle, denoted as Q_i.

Let f_i be the size of maximum hidden vertex set in S_i. Hence, for any $u, v \in S_i$, we have

$$\overline{uv} \subseteq P \Longleftrightarrow \overline{uv} \subseteq Q_i.$$

Thus, for any $X \subseteq S_i$, X is a hidden vertex set in P as long as it is a hidden vertex set in Q_i. Accordingly, it can be computed via the dynamic programming in Theorem 8.28, where we need to deal with at most $t_{i-1}^3 t_i^3$ proper pairs.

Therefore, to compute the f_i for all $i \in [c]$, the number of proper pairs we need to deal with is at most

$$\sum_{i=1}^{c} t_{i-1}^3 t_i^3 \leq \sum_{i=1}^{c} t_i^6.$$

Notice that $\sum_{i=1}^{c} t_i \leq 2n$ and $t_i \leq n$, then we have

$$\sum_{i=1}^{c} t_i^6 \leq (2n)^6.$$

Therefore, there are at most $O(n^6)$ proper pairs, and each of them takes $O(1)$ to compute, so the overall time complexity is $O(n^6)$.

Meanwhile, Let OPT be the maximum hidden vertex set of P, then we have

$$\sum_{i=1}^{c} |OPT \cap S_i| \geq 2|OPT|.$$

Thus, we have $\max_{i=1}^{c}\{f_i\} \geq \frac{2}{c}|OPT|$, which provides us with a solution that is at least $\frac{2}{c}$-competitive. □

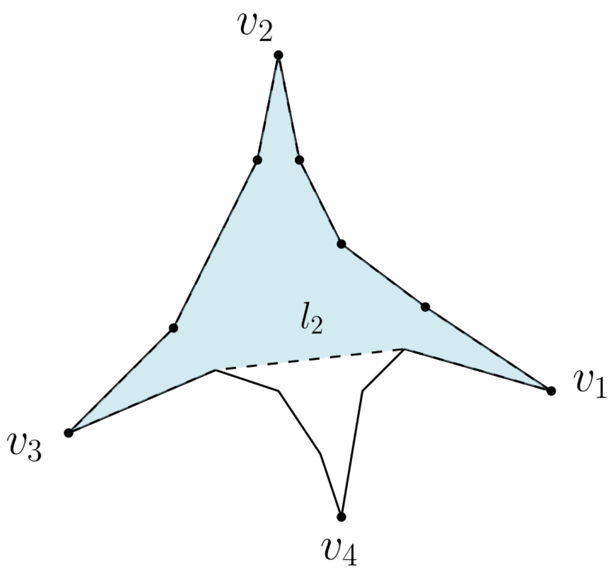


Figure 8.7: This figure illustrates the algorithm in Theorem 8.30, where $c = 4$. As is showed, l_2 is the geodesic path from v_3 to v_1, and Q_2 is the pseudotriangle by blue color which is filled.

Chapter 9

Fan-shaped Polygon and Terrain

In this chapter, we will show that both the maximum hidden vertex set and the maximum hidden point set of the fan-shaped polygon P and a terrain T can be solved efficiently in polynomial time. The algorithm for $\mathrm{HV}(P)$ is already argued about in [28], of which we will give an alternative proof. After that, we will propose our algorithm, by which hidden point problem in fan-shaped polygon and terrain is firstly resolved so far.

9.1 Terrain

Definition 9.1 *A terrain T is a polygonal chain $T = (p_0, p_1, \cdots, p_{n-1})$ such that it is strictly x-monotone, which means that $\forall i \in [0, n-2], x_{p_i} < x_{p_{i+1}}$.*

Further, p_0 and p_{n-1} is called the left and right endpoint of T.

Let u, v be two points such that $u \in T$ and $v \in T$, then u and v is considered to visible to each other if $\overline{uv}$ lies completely above T.

Definition 9.2 *Let T be a terrain, its visibility graph $VG(T)$ is defined as $VG(T) = (V, E)$, where V is the vertices of T, and E is the edges connecting the pair of vertices which are visible to each other.*

To characterize and recognize the visibility graph of terrain is considered an open problem. Up to now, no equivalent characterization of $VG(T)$ has already be discovered. Previously, there are two necessary properties proved in [2], in which they are together called ”persistence”.

Definition 9.3 *Let G be a graph with $(p_0, p_1, \cdots, p_{n-1})$ as a Hamiltonian path in G, then G is called a* ***persistent*** *graph if G satisfies the following two conditions:*

- ***X-property*** *For each tuple (a, b, c, d) such that $0 \leq a < b < c < d < n$, if $(p_a, p_c) \in G$ and $(p_b, p_d) \in G$, we have $(p_a, p_d) \in G$.*
- ***Bar-property*** *For each pair of $i, j \in [0, n-1]$ such that $i + 2 \leq j$, if $(p_i, p_j) \in G$, there exists $k \in [i+1, j-1]$ such that $(p_i, p_k) \in G$ and $(p_k, p_j) \in G$.*

Lemma 9.4 *Let T be a terrain, then $VG(T)$ is a persistent graph [2].*

Since the persistence property was proposed, it has been conjectured to also be the sufficient condition for a Hamiltonian graph G be a visibility graph of some terrain T for a long period. However, recently it has been shown that persistence is not enough in [8] by giving a persistent graph with 35 vertices, which is not the visibility graph of any terrain.

Similar to the polygon, based on the visibility, we can further define the hidden point set and hidden vertex set for a terrain.

Definition 9.5 *Let T be a terrain, and X be a set of points.*

X is called a hidden point set in T if $X \subseteq T$, and any two points in X is invisible to each other. The largest possible size of the hidden point set in T is denoted as $\mathrm{HP}(T)$.

X is called a hidden vertex set in T if $X \subseteq V(T)$, and any two points in X is invisible to each other. The largest possible size of the hidden vertex set in T is denoted as $\mathrm{HV}(T)$.

9.2 Relation between Terrain and Fan-shaped Polygon

In this section, we will show that the visibility graph of a terrain T is actually equivalent to the visibility graph of some fan-shaped polygon P, excluding the hub vertex, and vice versa.

Lemma 9.6 *Let $T = (p_1, p_2, \cdots, p_{n-1})$ be a terrain, then there exists p_0 such that $P = (p_0, p_1, \cdots, p_{n-1})$ is fan-shaped polygon with $p_0 \in hub(P)$ and p_0 lies above T.*

Proof We first explicitly construct such p_0 and then prove its correctness.

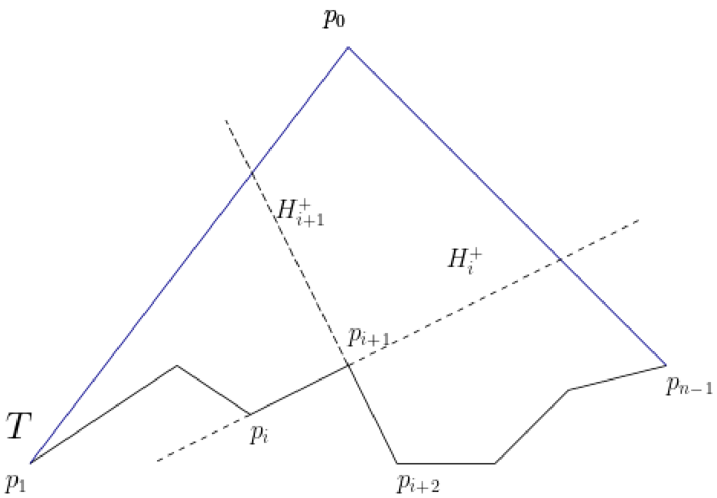


Figure 9.1: p_0 is the added hub vertex which transforms the terrain T into a fan-shaped polygon P.

For each $i \in [1, n-2]$, consider the close half-plane H_i^+ which is above the line $\overline{p_i p_{i+1}}$. Let $p_i = (x_i, y_i)$, then the equation of the line $\overline{p_i p_{i+1}}$ is given by

$$(x - x_i)(y_{i+1} - y_i) - (y - y_i)(x_{i+1} - x_i) = 0.$$

For any two distinct points a, b with $x_a \neq x_b$, denote the slope of $\overline{ab}$ as $k(a, b)$, and $k(a, b) = (y_a - y_b)(x_a - x_b)$. As $x_{i+1} - x_i > 0$, we have

$$H_i^+ = \{(x, y) | y \geq f_i(x) = k(p_i, p_{i+1})(x - x_i) + y_i\}.$$

Let $x_0 = (x_1 + x_{n-1})/2, y_0 = \max(\{f_i(x_0)\}_{i=1}^{n-2} \cup \{x_0, x_{n-1}\}) + d, (d > 0)$, then we get $\forall i \in [0, n-2], p_0 = (x_0, y_0) \in H_i^+$.

Let $P = (p_0, p_1, \cdots, p_{n-1})$, we will argue that $\forall u \in T, \overline{p_0 u} \subseteq P$, implying that P is a fan-shaped polygon and $p_0 \in hub(P)$.

For the sake of contraction, suppose that there exists $u \in T$ such that $\overline{p_0 u} \nsubseteq P$. Further, we assume that $x_u \leq x_0$. If $x_u = x_0$, since T is x-monotone, u is the unique intersection of T and $\overline{p_0 u}$. Otherwise, we have $x_u < x_0$.

Further, since $\overline{p_0 u} \nsubseteq P$, there exists point $v \neq u, v \in \overline{p_i p_{i+1}}$, such that $v = \overline{p_0 u} \cap \overline{p_i p_{i+1}}$. Then, there exists a vertex $p_t \in T[u, p_{n-1}]$ such that $x_{p_t} > x_u$ and p_t strictly lies above the line $\overline{p_0 u}$, and further suppose that p_t is the first vertex to do so.

The equation of $\overline{p_0 u}$ is given by $y = k(u, p_0)(x - x_u) + y_u$, notice that p_t is the first vertex above the line $\overline{p_0 u}$, thus we have

$$\begin{aligned} y_t &> k(u, p_0)(x_t - x_u) + y_u, \\ y_{t-1} &\leq k(u, p_0)(x_{t-1} - x_u) + y_u. \end{aligned}$$

Therefore, $k(p_{t-1}, p_t) > k(u, p_0) \geq k(p_0, p_{t-1})$. Given that $H^+_{t-1} = \{(x, y) | y \geq f_{t-1}(x)\}$, we have

$$\begin{aligned} f_{t-1}(x_0) &= k(p_{t-1}, p_t)(x_0 - x_{t-1}) + y_{t-1} \\ f_{t-1}(x_0) - y_0 &= k(p_{t-1}, p_t)(x_0 - x_{t-1}) - (y_0 - y_{t-1}) \\ &= (x_0 - x_t)(k(p_{t-1}, p_t) - k(p_0, p_{t-1})) \\ &> 0. \end{aligned}$$

Thus, we have $p_0 = (x_0, y_0) \notin H^+_{t-1}$, leading to a contradiction.

Therefore, $P = (p_0, p_1, \cdots, p_{n-1})$ is indeed a fan-shaped polygon and $p_0 \in hub(P)$. □

From Lemma 9.6, we can see that a terrain T can always be extended to a fan-shaped polygon P by connecting two endpoints to the introduced hub vertex p_0 above the terrain T. Then, naturally the following question comes out, given a fan-shaped polygon $P = (p_0, p_1, \cdots, p_{n-1})$ with $p_0 \in hub(P)$, is the polygonal chain $(p_1, \cdots, p_{n-1})$ always a terrain as well? The answer is negative, and as is shown in the following figure 9.2, we can not place an x-axis on the figure such that it is x-monotone.

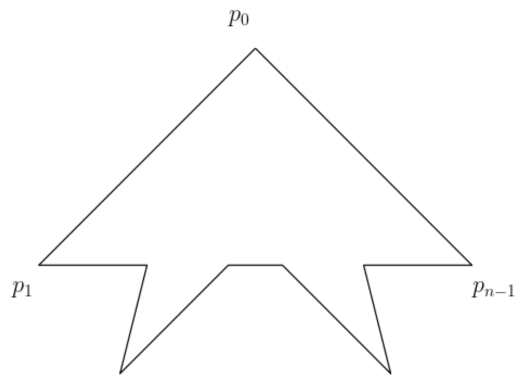


Figure 9.2: The polygonal chain $(p_1, \cdots, p_{n-1})$ is not a terrain

However, in terms of the visibility graph itself, we can see that $VG(P) \setminus p_0$ is actually identical to the visibility graph of some terrain T by this following lemma.

Lemma 9.7 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a fan-shaped polygon and $p_0 \in hub(P)$, then there exist a terrain $T = (q_1, \cdots, q_{n-1})$ and a continuous mapping $f : P(p_1, p_{n-1}) \to T$ such that:*

- $\forall i \in [n-1], f(p_i) = q_i$,
- $\forall u, v \in P(p_1, p_{n-1})$, $\overline{uv} \subseteq P$ *if and only if* $\overline{f(u)f(v)}$ *lies completely above the terrain T.*

The explicit construction is proposed in [45] by elevating the hub vertex p_0 into $\mathbf{R}^3$ and then projecting the remaining vertices through p_0 onto another plane. This implies that the visibility graph of a fan-shaped polygon, excluding the hub vertex, is indeed equivalent to the visibility graph of a terrain. Therefore, put Lemma 9.6 and Lemma 9.7 together, we can immediately give a characterization of $VG(P)$.

Lemma 9.8 *Let G be a simple graph, G is the visibility graph of some fan-shaped polygon P if and only if there exists a universal vertex u, which is adjacency to all the other vertices, such that $G \setminus u$ is the visibility graph of a terrain T.*

Remark 9.9 *This does not mean that we can already efficiently recognize the visibility graph of a fan-shaped polygon since we do not know how to deal with terrain itself yet.*

By the following lemmas, we can see that finding the maximum hidden vertex set or maximum hidden point set in a terrain, is indeed the same as in a fan-shaped polygon.

Lemma 9.10 *Let $T = (p_1, p_2, \cdots, p_{n-1})$ be terrain, $P = (p_0, p_1, \cdots, p_{n-1})$ be a fan-shaped polygon such that $p_0 \in hub(P)$ and p_0 lies above T. Then the following proposition stands.*

- *(a)* $\mathrm{HV}(P) = \mathrm{HV}(T)$*, and* $\exists\, X \subseteq V(T)$ *such that* $X = \mathrm{HV}(P)$*, and X is a hidden vertex set of both P and T.*
- *(b)* $\mathrm{HP}(P) = \mathrm{HP}(T)$*, and* $\exists\, X \subseteq T$ *such that* $X = \mathrm{HP}(P)$*, and X is a hidden point set of both P and T.*

Proof (a) is indeed trivial. Since $VG(P) \setminus p_0 = VG(T)$, and p_0 is a universal vertex in $VG(P)$, we have $\mathrm{HV}(P) = \mathrm{HV}(T)$. Meanwhile, $\{p_0\}$ can not be the unique maximum hidden vertex set, thus we can always find a maximum hidden vertex set X such that $p_0 \notin X$ and $X \subseteq V(T)$.

Then let us consider (b). Let $H = \{h_1, h_2, \cdots, h_k\}$ be a maximum hidden point set of P, and without loss of generality, we suppose that $p_0 \notin H$.

For each $i \in [k]$, let $l_i = \overrightarrow{p_0h_i} \cap T$. Since any point in P is visible to p_0, we have $\overline{p_0h_i} \subseteq P$, thus l_i is connected, and of course not empty. Further, let s_i be a point in l_i, and $S = \{s_i\}_{i=1}^{k}$.

Next we will argue that S is indeed a hidden point set in both P and T. For any $i \neq j$, it is clear that p_0, h_i and h_j are not collinear, and denote the triangle $conv(\{p_0, h_i, h_j\})$ as $C_{i,j}$. Since $\overline{p_0h_i} \subseteq P$, $\overline{p_0h_j} \subseteq P$, and $\overline{h_ih_j} \not\subseteq P$, there exists a vertex $r \in V(T)$, such that $r \in int(C_{i,j})$, and r lies strictly above $\overline{h_ih_j}$. Hence, notice that $\overline{p_0h_i} \subseteq \overline{p_0s_i}$ and $\overline{p_0h_i} \subseteq \overline{p_0s_i}$, we have $r \in int(conv(\{p_0, s_i, s_j\}))$, and r also lies strictly above $\overline{s_is_j}$. Therefore, $\overline{s_is_j} \not\subseteq P$, and $\overline{s_is_j}$ does not lie completely above T. This indicates that S is indeed a hidden point set with regard to both P and T. $\square$

Accordingly, finding the maximum hidden point/vertex set in a terrain $T = (p_1, \cdots, p_{n-1})$ could take the following three steps.

1. Find the vertex p_0 such that $P = (p_0, p_1, \cdots, p_{n-1})$ is a fan-shaped polygon, and $p_0 \in hub(P)$.
2. Figure out the maximum hidden point/vertex set S in P.
3. Relocate S to S' such that $S' \subseteq T$, and S' is a hidden set with regard to T.

The procedure in step 3 could follow the proof of Lemma 9.10. Therefore, as long as we solve these problems in the fan-shaped polygon, we immediately solve them in the terrain as well.

9.3 Hidden Vertices in Fan-shaped Polygon and Terrain

Lemma 9.11 *Let $P = (p_0, p_1, \cdots, p_{n-1})$. Let $p_i, p_j, i < j$ be two vertices such that $I_P(p_i, p_j) = 0$ and k be the smallest integer in the interval $[i, j]$ such that $I_P(p_k, p_j) = 1$. Then $\forall u \in [i, k-1], v \in [k+1, j]$, we have $I_P(p_u, p_v) = 0$, which means they are invisible to each other.*

Proof Suppose the contradiction that there exist $u \in [i, k-1], v \in [k+1, j]$ such that $I_P(p_u, p_v) = 1$. Since the visibility graph $VG(P) \setminus \{p_0\}$ is persistent, it satisfies the **X-property**. Notice that p_j is invisible to any vertex p_u with $u < k$, and then we have $v < j$. Therefore, consider the tuple (u, k, v, j) with $u < k < v < j$, give that $(p_u, p_v) \in VG(P)$ and $(p_k, p_j) \in VG(P)$, by Definition 9.3, we have $(p_u, p_j) \in VG(P)$, which violates the minimality of the integer k. $\square$

Definition 9.12 *Let* $P = (p_0, p_1, p_2, \cdots, p_{n-1})$ *be a fan-shaped polygon and* $p_0 \in hub(P)$*, then the subfan* $sub(P, l, r)$ *is defined by the following:*

$$sub(P, l, r) := (p_0, p_l, p_{l+1}, \cdots, p_r).$$

Lemma 9.13 *Let* $P = (p_0, p_2, p_2, \cdots, p_{n-1})$ *be a fan-shaped polygon and* $p_0 \in hub(P)$*, then we can compute* $\{\mathrm{HV}(sub(P, l, r)) | \forall l \leq r\}$ *in time complexity* $O(n^2)$ *by dynamic programming.*

Proof We first sketch the dynamic programming, then prove its correctness and efficiency.

Let $f(i, j)$ denote the size of maximum hidden vertex set for the subfan $sub(P, i, j)$. It is clear that since p_0 is visible to any vertex, p_0 itself will never become the unique optimal solution. Therefore, we only need to take vertices $p_i, p_{i+1}, \cdots, p_j$ into consideration.

If $i = j$, the optimal solution is to pick the vertex p_i, and $f(i, j) = 1$ clearly. Otherwise, we could compute $f(i, j)$ by the following formula,

$$f(i,j) = \begin{cases} \max\{f(i+1,j), f(i,j-1)\}, & I_P(p_i, p_j) = 1, \\ \max\{f(i+1,j), f(i,j-1), f(i,k-1) + f(k+1,j)\} & I_P(p_i, p_j) = 0. \end{cases}$$

where k is the smallest integer in the interval $[i, j]$ such that $I_P(p_k, p_j) = 1$.

We prove this by induction on the length of interval $[i, j]$.

Let $m = j - i + 1$. If $m = 1$, we have $i = j$ and the optimal solution $f(i, j) = 1$ holds clearly.

Suppose that $\forall m = j - i + 1 \leq l$, $f(i, j) = \mathrm{HV}(sub(P, i, j))$. Consider the case $m = l + 1$.

Let A be the maximum hidden vertex set for $sub(P, i, j)$. There are in total three cases.

- $p_i \notin A$, then A is actually the optimal solution for $sub(P, i + 1, j)$, $|A| = \mathrm{HV}(sub(P, i + 1, j)) = f(i + 1, j)$.
- $p_j \notin A$, then A is actually the optimal solution for $sub(P, i, j - 1)$, $|A| = \mathrm{HV}(sub(P, i, j - 1)) = f(i, j - 1)$.
- $p_i \in A \wedge p_j \in A$, this can only take place when $I_P(p_i, p_j) = 0$. Notice that we have p_k and p_j are visible to each other, then $p_k \notin A$. Hence, by Lemma 9.11, we have $\{p_i, p_{i+1}, \cdots, p_{k-1}\}$ and $\{p_{k+1}, p_{k+2}, \cdots, p_j\}$ are invisible to each other. Therefore,

A is composed by the maximum hidden vertex set of subfan $sub(P,i,k-1)$ and subfan $sub(P,k+1,j)$, $|A| = \mathrm{HV}(sub(P,i,k-1)) + \mathrm{HV}(sub(P,k+1,j)) = f(i,k-1) + f(k+1,j)$.

Thus, the induction is completed.

Further, the number of pairs (i,j) is in $O(n^2)$, and each $f(i,j)$ takes $O(1)$ time to compute. Meanwhile, note that computing k for all $(i<j)$ also takes $O(n^2)$. Therefore, the overall time complexity is $O(n^2)$. □

Corollary 9.14 *Let $T = (p_0, p_1, \cdots, p_{n-1})$ be terrain with n vertices, then we can compute $\mathrm{HV}(T(p_l, p_r))$ for all $l \le r$ in time complexity $O(n^2)$ by dynamic programming.*

Proof This corollary is implied by Lemma 9.6 and Lemma 9.13. □

Corollary 9.15 *Let P be fan-shaped polygon, and $S \subseteq V(P)$ be a subset of vertices. The maximum hidden vertex set in S can be computed in $O(n^2)$ time.*

Proof This can be achieved by adapting the dynamic programming in Lemma 9.13, by avoiding any vertex in $V(P) \setminus S$ in the optimal solution of any subproblem. □

9.4 Hidden Points in Fan-shaped Polygon

In this section, we will introduce how to efficiently compute the maximum hidden points in fan-shaped polygon and terrain.

9.4.1 Partially Ordered Set

Definition 9.16 *A partially ordered set (poset) $\mathcal{P}$ is a pair $\mathcal{P} = (X, \preccurlyeq)$, where X is the set of elements and $\preccurlyeq$ is a binary relation (partial order) on X satisfying the following properties:*

- ***Reflexivity**: $\forall x \in X$, $x \preccurlyeq x$.*
- ***Antisymmetry**: $\forall x, y \in X$, if $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$.*
- ***Transitivity**: $\forall x, y, z \in X$, if $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$.*

Further, we denote $x \prec y$ if $x \preccurlyeq y$ and $x \neq y$. Besides, we write $x \succ y$ and $y \succ x$ if neither $x \preccurlyeq y$ nor $y \preccurlyeq z$.

Definition 9.17 *Let $\mathcal{P} = (X, \preccurlyeq)$ be a partially ordered set. A ordered list of elements $L = (x_1, x_2, \cdots, x_k)$, is call a chain if $\forall\ i \in [k-1]$, we have $x_i \preccurlyeq x_{i+1}$.*

Definition 9.18 *Let $\mathcal{P} = (X, \preccurlyeq)$ be a partially ordered set, and $C \subseteq X$ be a subset of elements. C is called an antichain if $\forall x, y \in C, x \neq y$, we have $x \succ y$ and $y \succ x$.*

Further, the largest antichain of $\mathcal{P}$ is the antichain in $\mathcal{P}$ with largest possible number of elements, and the its size is denoted as $\mu(\mathcal{P})$.

Indeed, a chain is a subset of elements such that the partial order $\preccurlyeq$ restricted on it becomes a total order, and an antichain is a subset of elements where each two of them are not comparable. For instance, let X be the set of positive integers and $|$ be the binary relation such that $a|b$ if and only if a is a divisor of b. Thus, all the powers of 2 compose a chain and all the primes form an antichain.

Definition 9.19 *Let $\mathcal{P} = (X, \preccurlyeq)$ be a finite partially ordered set. A chain decomposition of $\mathcal{P}$ is a collection of chains $\{C_1, C_2, \cdots, C_k\}$ such that*

$$C_i \cap C_j = \varnothing, \forall\, i \neq j,$$
$$X = \bigcup_{i=1}^{k} C_i.$$

Further, the minimum chain decomposition of $\mathcal{P}$ is such collection C with the smallest number of chains, and we denote its size as $\nu(\mathcal{P})$.

Based the above preliminary, we could introduce the most fundamental theorem of the partially ordered set.

Theorem 9.20 ***Dilworth's Theorem*** *: Let $\mathcal{P} = (X, \preccurlyeq)$ be a partially ordered set, then the minimum chain decomposition of $\mathcal{P}$ has the same size of the largest antichain of $\mathcal{P}$, which means $\nu(\mathcal{P}) = \mu(\mathcal{P})$.*

Proof This is proved in [18]. □

Let us make an example on the inclusion, which is a partial order on sets. Let $U = \{x, y, z\}$ be the ground set. Let X be the collection of all the subsets of U, $\preccurlyeq$ be the relation such that $A \preccurlyeq B$ if $A \subseteq B$, and $\mathcal{P} = (X, \preccurlyeq)$ be the partially ordered set. $\mathcal{P}$ is illustrated as the following figure 9.3.

Accordingly, one of the largest antichain of $\mathcal{P}$ is $A = \{\{x\}, \{y\}, \{z\}\}$, and one of the minimum chain decomposition of $\mathcal{P}$ is $C = \{L_1, L_2, L_3\}$, where $L_1 = \{\varnothing, \{x\}, \{x, y\}, \{x, y, z\}\}$, $L_2 = \{\{y\}, \{y, z\}\}$, and $L_3 = \{\{z\}, \{x, z\}\}$. Indeed, they have the same size $|A| = |C| = 3$.

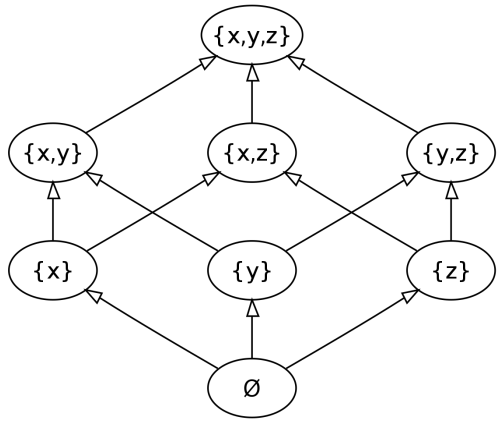


Figure 9.3: $A \preccurlyeq B$ if and only if there is a directed path routed from A to B in the graph.

9.4.2 Poset in Fan-shaped Polygon

In this subsection, we could define a partially ordered set on the edges of fan-shaped polygon. Before that, we need to discuss some basic propositions about the visibility between the edges.

Lemma 9.21 *Let $P = (p_0, p_1, \cdots, p_n)$ be a simple polygon, and $e_i = (p_i, p_{i+1})$, $e_j = (p_j, p_{j+1})$ be two edges, then the following three propositions are equivalent:*

- *(a) $\forall\ u \in \{p_i, p_{i+1}\}, v \in \{p_j, p_{j+1}\}, I_P(u, v) = 1$.*
- *(b) $\forall\ u \in \overline{p_i p_{i+1}}, v \in \overline{p_j p_{j+1}}, I_P(u, v) = 1$.*
- *(c) $X = \{p_i, p_{i+1}\} \cup \{p_j, p_{j+1}\}$, vertices in X are in convex position and $conv(X) \subseteq P$.*

Proof (a) $\Longrightarrow$ (c): By Lemma 6.9, since $X = \{p_i, p_{i+1}\} \cup \{p_j, p_{j+1}\}$ is a clique in $VG(P)$, then vertices in X are in convex position and $conv(X) \subseteq P$.

(c) $\Longrightarrow$ (b): $\forall u \in \overline{p_i p_{i+1}}, v \in \overline{p_j p_{j+1}}$, we have $\overline{uv} \subseteq conv(X) \subseteq P$, indicating that $I_P(u, v) = 1$.

(b) $\Longrightarrow$ (a): This is trivial. □

Then we are ready to define the partial order on the edges of the fan-shaped polygon.

Lemma 9.22 *Let $P = (p_0, p_1, \cdots, p_n)$ be a fan-shaped polygon, and $p_0 \in hub(P)$. Let $e_i = (p_i, p_{i+1}), e_j = (p_j, p_{j+1})$ be two edges of P.*

Let $\preccurlyeq$ be a binary relation. For $i \in [n-2], j \in [n-2]$, $e_i \preccurlyeq e_j$ if and only if

$$[i \leq j] \wedge [\forall u \in e_i, v \in e_j, I_P(u, v) = 1].$$

Then, $\preccurlyeq$ is indeed a partial order.

Proof The reflexivity and antisymmetry of $\preccurlyeq$ is clear to hold, and we will prove the transitivity of $\preccurlyeq$.

Let e_i, e_j, e_k be three edges such that $e_i \preccurlyeq e_j$ and $e_j \preccurlyeq e_k$, and we will argue that $e_i \preccurlyeq e_k$.

Without loss of generality, we assume that e_i, e_j and e_k are disjoint, which means they do not share any common vertex[1]. Specifically, we have $i+1 < j$ and $j+1 < k$.

Consider any $u \in \{p_i, p_{i+1}\}$ and $v \in \{p_k, p_{k+1}\}$. Since $e_i \preccurlyeq e_j$, we have $I_P(u, p_{j+1}) = 1$. Similarly, since $e_j \preccurlyeq e_k$, we have $I_P(p_j, v) = 1$.

Notice that u, p_j, p_{j+1}, v are in order, and $I_P(u, p_{j+1}) = I_P(p_j, v) = 1$. By the X-property in Definition 9.3, we know that $I_P(u, v) = 1$.

Therefore, $\forall u \in \{p_i, p_{i+1}\}, v \in \{p_k, p_{k+1}\}$, we have $I_P(u, v) = 1$. By Lemma 9.21, we know that this is equivalent to $e_i \preccurlyeq e_k$, thus justifying the transitivity and completing the proof. $\square$

According to this partial order, we can further define the partially order set in a fan-shaped polygon.

Definition 9.23 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be fan-shaped polygon and $p_0 \in hub(P)$.*

Then, the visibility partially ordered set of P, denoted as $VP(P)$, is defined as $VP(P) := (X, \preccurlyeq)$, where $X = \{e_i = (p_i, p_{i+1}) | i \in [n-2])\}$, and $e_i \preccurlyeq e_j$ if and only if

$$[i \leq j] \wedge [\forall u \in e_i, v \in e_j, I_P(u, v) = 1].$$

In the following lemmas, we will show that this poset is closely related to the hidden point set and convex covering of the fan-shaped polygon.

Lemma 9.24 *Let P be a fan-shaped polygon, and $\mathcal{P} = VP(P)$ be its visibility partially ordered set. Then,* $\textsc{Cover}(P) \leq \nu(\mathcal{P})$.

Proof Let C be a chain in $\mathcal{P}$, and $C = (l_1, l_2, \cdots, l_m)$, where $l_i = e_{t_i} = (p_{t_i}, p_{t_i+1}), t_i \in [n-2]$.

Let $X_C = \bigcup_{i=1}^{m} \{p_{t_i}, p_{t_i+1}\}$. Since C is a chain in $\mathcal{P}$, vertices in X_C are pairwisely visible, and thus X_C is a clique in $VG(P)$. Further, since p_0

[1]We can not say the arguments for the other cases are the same, but they are indeed easier to deal with.

is visible to every vertex in X_C, $X_C \cup \{p_0\}$ is also a clique in $VG(P)$. Let $Y_C = X_C \cup \{p_0\}$. By Lemma 6.9, vertices in Y_C are in convex position, and $conv(Y_C) \subseteq P$. Denote the $conv(Y_C)$ as $f(C)$, which is the convex hull induced by C.

Let $\mathcal{C} = \{C_1, C_2, \cdots, C_k\}$ be a minimum chain decomposition of $\mathcal{P}$, and $k = \nu(P)$. For each $i \in [k]$, let $Q_i = f(C_i)$ be the convex hull induced by C_i. Hence, for each $i \in [n-2]$, there exists $j \in [k]$, such that $e_i \in C_j$, thus the triangle $(p_0, p_i, p_{i+1}) \subseteq Q_j$.

Notice that $P = \bigcup_{i=1}^{n-2}(p_0, p_i, p_{i+1})$, we have $P = \bigcup_{i=1}^{k} Q_i$, implying that $\mathcal{Q} = \{Q_1, Q_2, \cdots, Q_k\}$ is a convex cover of P and $\textsc{Cover}(P) \leq \nu(\mathcal{P})$. □

Lemma 9.25 *Let P be a fan-shaped polygon, and $\mathcal{P} = VP(P)$ be its visibility partially ordered set. Then,* $\mathrm{HP}(P) \leq \mu(\mathcal{P})$.

Proof Let $H = \{h_1, h_2, \cdots, h_k\}$ be a maximum hidden point set in H, and $k = \mathrm{HP}(P)$. By Lemma 9.10, we can further assume that $H \subseteq P(p_1, p_{n-1})$. For each $i \in [k]$, let $s_i = (p_{t_i}, p_{t_i+1})$ be the edge including the point h_i (if there are multiple ones then choose any of them).

Then, we can see that for any $i \neq j$, we have neither $s_i \preccurlyeq s_j$ nor $s_j \preccurlyeq s_i$. Otherwise, h_i and h_j should be visible to each other. Therefore, we can see that $\{s_i\}_{i=1}^{k}$ is indeed an antichain in $\mathcal{P}$, and thus $\mathrm{HP}(P) \leq \mu(\mathcal{P})$. □

Remark 9.26 *This can be induced immediately by Lemma 9.24, since* $\mathrm{HP}(P) \leq \textsc{Cover}(P)$ *and* $\mu(\mathcal{P}) = \nu(\mathcal{P})$.

9.4.3 Find Maximum Hidden Point Set in $O(n^2)$

In this subsection, we present the algorithm to find the maximum hidden point set of a fan-shaped polygon in $O(n^2)$.

Overview of the algorithm. Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a fan-shaped polygon, and $p_0 \in hub(P)$. For each edge $e_i = (p_i, p_{i+1}) \in [n-2]$, we locate two points $\{\alpha_i, \beta_i\} \subseteq e_i$ such that, if vertex p_i is invisible to vertex p_j, then we have $\forall u \in \{\beta_{i-1}, \alpha_i\}, \forall v \in \{\beta_{j-1}, \alpha_j\}$, u is invisible to v. Intuitively speaking, we are trying to arrange α_i sufficiently close to p_i, and β_i sufficiently close to p_{i+1}. And of course, we will show that such $\{\alpha_i\}_{i=1}^{n-2}$ and $\{\beta_i\}_{i=1}^{n-2}$ can be computed in polynomial time. After that, we will show that among these candidates $\{\alpha_i\}_{i=1}^{n-2} \cup \{\beta_i\}_{i=1}^{n-2}$, we can find a hidden point set, which is indeed the maximum hidden point set of P. Such technique can also be utilized to present a constant factor approximation for maximum hidden point set in a simple polygon [10].

First, we present the explicit construction of the set $\{\alpha_i\}_{i=1}^{n-2}$ and $\{\beta_i\}_{i=1}^{n-2}$ in the following definition.

Definition 9.27 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a fan-shaped polygon, and $p_0 \in hub(P)$. Denote $(p_i, p_{i+1}) \in E(P)$ as e_i.*

For each $i \in [n-1]$, denote Q_i as the set of Steiner points in $Vis_P(p_i)$.

For any $0 < i < j < n$, let $path_{i,j}$ denote the geodesic path from vertex p_i to p_j, and $inter_{i,j}$ be the set of intermediate vertices of $path_{i,j}$.

$$inter_{i,j} := V(path_{i,j}) \setminus \{p_i, p_j\}.$$

Let $l_{i,j}, 0 < i < j < n$, be defined as the following,

$$l_{i,j} := \begin{cases} line(u, \overrightarrow{p_i p_j}), & inter_{i,j} = \{u\}, \\ \varnothing, & otherwise, \end{cases}$$

where $line(u, \overrightarrow{v})$ denote the the line passing u and parallel to $\overrightarrow{v}$.

We define $Y_i, i \in [n-2]$ as the following,

$$\begin{aligned} Y_i := & \bigcup_{j=1}^{n-1} e_i \cap Q_j \\ & \bigcup_{j<i} int(e_i) \cap l_{j,i} \\ & \bigcup_{j>i} int(e_i) \cap l_{i+1,j}. \end{aligned}$$

Accordingly, let $\alpha_i \in int(e_i)$ such that $\overline{p_i \alpha_i} \cap Y_i = \varnothing$, and $\beta_i \in int(e_i)$ such that $\overline{\beta_i p_{i+1}} \cap Y_i = \varnothing$.

Then, we will prove that such explicit construction indeed preserves the invisibility based on the following propositions.

Proposition 9.28 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a fan-shaped polygon, and $p_0 \in hub(P)$. Let $e = (u, v)$ be an edge of P, r be a point in P, and $e \cap Vis(r) \neq \varnothing$. Then we have either $u \in Vis(r)$ or $v \in Vis(r)$.*

Proof We prove this by contradiction. Suppose the opposite that there exists a point r and an edge $e = (u, v)$ such that $u \notin Vis(r)$, $v \notin Vis(u)$, and $Vis(u) \cap \overline{uv} \neq \varnothing$.

Notice that $Vis(r) \cap \overline{uv}$ is connected since P is a simple polygon. Therefore, there exists $s \in \overline{uv}$ and $t \in \overline{uv}$ such that $Vis(u) \cap \overline{uv} = \overline{st}$. We further assume that u, s, t, v are in counter-clockwise order.

Since s is a Steiner point of $Vis(r)$, there exists a reflex vertex p_i, and $p_i \in int(\overline{rs})$. Similarly, there exists a reflex vertex p_j and $p_j \in int(\overline{rt})$.

Let $H^+_{s,r}$ and $H^+_{r,t}$ be the open half-planes on the left hand side of $\overrightarrow{sr}$ and $\overrightarrow{rt}$ respectively. Clearly, we can see that $u \in H^+_{s,r}$ and $v \in H^+_{r,t}$.

Now let's consider the location of p_0. Indeed, we have $p_0 \notin H^+_{s,r}$, otherwise p_0 is invisible to either r or u, contradicting that $p_0 \in hub(P)$. Similarly, we have $p_0 \notin H^+_{r,t}$. Meanwhile, $\overline{rp_0}$ can not properly intersect with $\overline{uv}$. Therefore, we have p_0 in the triangle (r, s, t) and thus further $p_0 \in conv(\{r, u, v\})$.

However, since $p_0 \in hub(P)$, we know that $\overline{p_0 r} \subseteq P, \overline{p_0 s} \subseteq P$ and $\overline{p_0 t} \subseteq P$. Since $p_0 \in int(conv(\{r, u, v\}))$, it is impossible to include these three vertices in a convex interior angle at vertex p_0, leading to the contradiction.

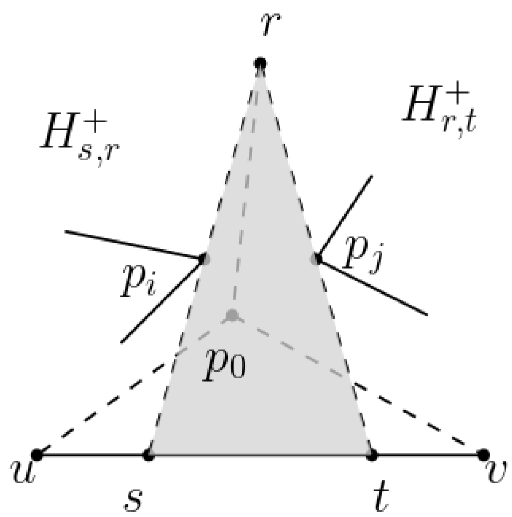


Figure 9.4: This figure illustrates the proof of Proposition 9.28.

Corollary 9.29 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a fan-shaped polygon. Let $e = (u, v)$ be an edge of P, r be a point in P. If r is invisible to both u and v, then $\forall w \in \overline{uv}$, r is invisible to w as well.*

Proof This is the contrapositive of Proposition 9.28. □

Proposition 9.30 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a fan-shaped polygon, and $p_0 \in hub(P)$. Let $\{\alpha_i\}_{i=1}^{n-2}$ and $\{\beta_i\}_{i=1}^{n-2}$ follow their definitions in Definition 9.27.*

Let p_i, p_j be two vertices in P and $I_P(p_i, p_j) = 0$, then we have $I_P(\beta_{i-1}, p_j) = 0$ and $I_P(\alpha_i, p_j) = 0$.

Proof We only prove this for α_i, as the argument for β_{i-1} is similar.

Let $e_i = (p_i, p_j)$, and consider $Vis(p_j) \cap e_i$. By Proposition 9.28, since $I_P(p_i, p_j) = 0$, there are two different cases.

- $Vis(p_j) \cap e_i = \varnothing$, then of course $\alpha_i \notin Vis(p_j)$, and $I_P(\alpha_i, p_j) = 0$.
- $Vis(p_j) \cap e_i \neq \varnothing$, then there exists point $r \in int(e_i)$ such that $Vis(p_j) \cap e_i = \overline{rp_{i+1}}$, where r is a Steiner point of $Vis(p_j)$. But, by our construction of α_i, we have $r \notin \overline{p_i\alpha_i}$, and $\overline{p_i\alpha_i} \cap \overline{rp_{i+1}} = \varnothing$, implying that $I_P(\alpha_i, p_j) = 0$. □

Lemma 9.31 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a fan-shaped polygon, and $p_0 \in hub(P)$. Let $\{\alpha_i\}_{i=1}^{n-2}$ and $\{\beta_i\}_{i=1}^{n-2}$ follow their definitions in Definition 9.27.*

Let p_i, p_j be two vertices of P, and $I_P(p_i, p_j) = 0$, then we have $\forall u \in \{\beta_{i-1}, \alpha_i\}, \forall v \in \{\beta_{j-1}, \alpha_j\}$, $I_P(u, v) = 0$.

Proof Let $S = \{\alpha_i\}_{i=1}^{n-2} \cup \{\beta_i\}_{i=1}^{n-2}$, and $T = V(P(p_1, p_{n-1}))$. We define $f : S \to T$ such that

$$\forall i \in [n-2], f(\alpha_i) = p_i, f(\beta_i) = p_{i+1}.$$

Further, we say that α_i is associated to vertex p_i, and β_i is associated to vertex p_{i+1}. Intuitively, by saying u associated to v, we indicates that u is very close to v and their visible areas are quite similar, though by definition v is not necessarily the vertex closest to point u.

We will prove that $\forall u, v \in S$, if $f(u)$ is invisible to $f(v)$, then u is also invisible to v. In this way, (u, v) preserves the invisibility of $(f(u), f(v))$.

Suppose that $(a, b), (c, d)$ be two edges of $P(p_1, p_{n-1})$ such that

$$u \in (a, b), f(u) = b, v \in (c, d), f(v) = c.$$

Without loss of generality, we assume that $\{a, b\} \cap \{c, d\} = \varnothing$, and (a, b) appears before (c, d) in the polygonal chain $P(p_1, p_{n-1})$. By our definition, we have b is invisible to c.

If b is invisible to both c and d, then by Proposition 9.30, we have u is also invisible to both c and d. By Corollary 9.29, we know that u is indeed invisible to v.

Similarly, if c is invisible to both a and b, we also have u is indeed invisible to v.

Therefore, we could further assume that $I_P(a,c) = I_P(b,d) = 1$. Then, let's consider the order of the vertex $\{a,b,c,d\}$ in P, which should be in counter-clockwise order. There are four different cases to consider.

- (b,a,d,c). This case is actually impossible. Since $I_P(a,c) = I_P(b,d) = 1$, by the X-property in Definition 9.3, we have $I_P(b,d) = 1$, contradicting the assumption.
- (b,a,c,d). In this case, $C = (b,a,c,d)$ is an ordered cycle of length four in $VG(P)$. By Lemma 2.25, C has at least one chord. By assumption, (b,c) can not be a chord, and then $I_P(a,d) = 1$. In this case, the general position of them is shown in the following figure, where a is in the interior of the triangle (b,c,d).

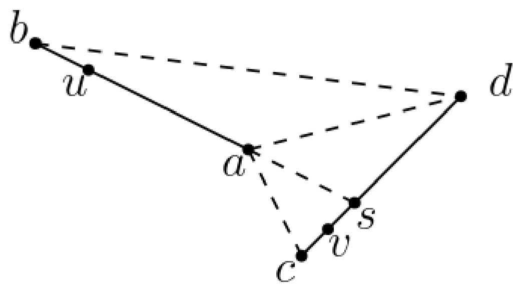


Figure 9.5: This figure illustrates the case of (b,a,c,d).

  Clearly, we can see that there exist $s \in int(\overline{cd})$, and s is a Steiner point of $Vis(b)$. Thus, by our construction, we have $v \in int(\overline{cs})$, implying that $I_P(u,v) = 0$.
- (a,b,d,c). This case is similar to the case (b,a,c,d) so we omit the proof here.
- (a,b,c,d). This case is the most complicated one, and also the only reason why we need to introduce $\{l_{i,j}\}_{i<j}$ in Definition 9.27.

  Let $b = p_i$ and $c = p_j$ with $i < j$. Let $path_{i,j}$ and $l_{i,j}$ follow their definitions in Definition 9.27.

Since $\overline{p_i p_j} \not\subseteq P$, $path_{i,j}$ is a polygonal chain in P with at least three vertices. By Theorem 1 in [28], $path_{i,j}$ is indeed a reflex chain, and $V(path_{i,j}) \subseteq V(P(p_i, p_j))$. Thus, let $path_{i,j} = \{p_i, w_1, \cdots, w_k, p_j\}$.

If $k = 1$, then $l_{i,j} = line(w_1, \overrightarrow{p_i p_j})$. Since p_i, p_j, u, v all lie on the same side of $l_{i,j}$, we have (u, w_1, v) is a reflex chain in P (if we consider u and v as the vertices of P). Thus, we have $I_P(u, v) = 0$.

If $k > 1$, we have $I_P(p_i, w_2) = I_P(w_{k-1}, p_j) = 0$. Further, by Proposition 9.30, we have $I_P(u, w_2) = I_P(w_{k-1}, v) = 0$. In fact, $C = (u, w_1, \cdots, w_k, v)$ is also a reflex chain in P, if we consider u and v as the vertices of P. Thus, we have $I_P(u, v) = 0$.

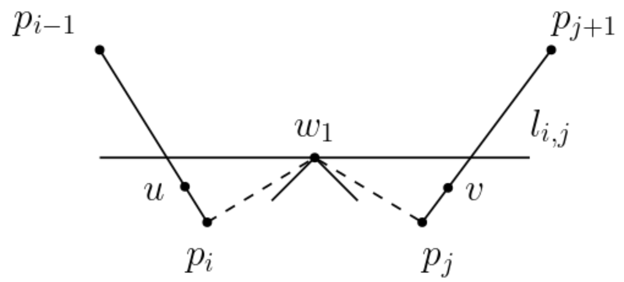


Figure 9.6: This illustrates the proof of the case (a, b, c, d).

Therefore, we conclude that $\forall u \in S, v \in S$, as long as $I_P(f(u), f(v)) = 0$, we have $I_P(u, v) = 0$, thus completing the proof. $\square$

By the last lemma, we can see that such construction indeed preserves the invisibility. Actually, in the next lemma, we show that what we achieved is much more than that. Instead of merely preserving the invisibility, we are also creating the invisibility at the same time.

Lemma 9.32 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a fan-shaped polygon, and $p_0 \in hub(P)$. Let $\mathcal{P} = (X, \preccurlyeq) = VP(P)$ be the visibility partially ordered set of P. Let $\{\alpha_i\}_{i=1}^{n-2}$ and $\{\beta_i\}_{i=1}^{n-2}$ follow their definitions in Definition 9.27.*

Let $e_i = (p_i, p_{i+1}), e_j = (p_j, p_{j+1})$ be two edges of P, and $i < j$. If $\{e_i, e_j\}$ is an antichain in $\mathcal{P}$, then $I_P(\beta_i, \alpha_j) = 0$.

Proof We first consider the trivial case, in which $j = i + 1$. Since $\{e_i, e_{i+1}\}$ is a antichain, we have p_{i+1} is a reflex vertex in P, indicating that $I_P(\beta_i, \alpha_{i+1}) = 0$.

Other than the trivial case, we always have $i + 1 < j$, and $e_i \cap e_j = \varnothing$. If $I_P(p_{i+1}, p_j) = 0$, by Lemma 9.31, we already have $I_P(\beta_i, \alpha_j) = 0$.

Otherwise, suppose that $I_P(p_{i+1}, p_j) = 1$. Accordingly, let $L = (p_i, p_{i+1}, p_j, p_{j+1})$ be the polygonal chain, and $L \subseteq P$. Note that L can not be a convex chain in P, otherwise we would have $e_i \preccurlyeq e_j$. Therefore, either (p_i, p_{i+1}, p_j) is a reflex chain or (p_{i+1}, p_j, p_{j+1}) is a reflex chain.

Without loss of generality, suppose that (p_i, p_{i+1}, p_j) is a reflex chain. We can see that this is indeed the second case we already dealt with in the proof of Lemma 9.31. Similar to that, we can see that $I_P(\beta_i, \alpha_j) = 0$.

Therefore, we conclude that if $\{e_i, e_j\}$ is an antichain, we always have $I_P(\beta_i, \alpha_j) = 0$. □

The reason why this creates the invisibility is that in Lemma 9.32, p_i and p_j do not necessarily to be invisible. As long as $\{e_i, e_j\}$ is an antichain in $\mathcal{P}$, we always have $I_P(\beta_i, \alpha_j) = 0$.

Lemma 9.33 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a fan-shaped polygon, and $p_0 \in hub(P)$. Let $\mathcal{P} = (X, \preccurlyeq) = VP(P)$ be the visibility partially ordered set of P.*

Let $L = \{l_1, l_2, \cdots, l_k\}$ be an antichain in $\mathcal{P}$, then we can find a hidden point set $H = \{h_1, h_2, \cdots, h_k\}$ in P, such that $\forall i \in [k], h_i \in l_i$.

Proof Let $l_i = e_{t_i} = (p_{t_i}, p_{t_i+1})$, and $\forall i \in [k-1]$, we assume $t_i < t_{i+1}$.

Let $a_i = \alpha_{t_i}$, and $b_i = \beta_{t_i}$. We prove the following statement by induction: Let $M = \{b_1\} \cup (\bigcup_{i=2}^{k-1}\{a_i, b_i\}) \cup \{a_k\}$. There exists $H \subseteq M$, such that H is a hidden point set, and $|H| = k$.

For the case $k \leq 2$, it is trivial. Suppose that $\forall k \leq m$, the statement holds, and consider the case for $k = m + 1$.

Similar to Lemma 9.32, we know that $I_P(b_1, a_{m+1}) = 0$. By Lemma 9.11, there exists a blocking vertex $p_j \in P(b_1, a_{m+1})$ such that $\forall u \in P(b_1, p_j) \setminus \{p_j\}, v \in P(p_j, a_{m+1}) \setminus \{p_j\}, I_P(u, v) = 0$.

Let m' be the largest integer in $[m+1]$ such that $t_{m'} < j$. Let $M_1 = \{b_1\} \cup (\bigcup_{i=2}^{m'-1}\{a_i, b_i\}) \cup \{a_{m'}\}$ and $M_2 = \{b_{m'+1}\} \cup (\bigcup_{i=m'+2}^{m}\{a_i, b_i\}) \cup \{a_{m+1}\}$.

By induction, there exists a hidden point set $H_1 \subseteq M_1, |H_1| = m'$ and another hidden point set $H_2 \subseteq M_2, |H_2| = m - m' + 1$. Hence, $\forall u \in H_1, v \in H_2$, we have $u \in P(b_1, p_j) \setminus \{p_j\}, v \in P(p_j, a_{m+1}) \setminus \{p_j\}$, thus $I_P(u, v) = 0$.

Therefore, $H = H_1 \cup H_2$ is indeed a hidden point set, and $|H| = m + 1$, thus completing the induction. □

Corollary 9.34 *Let $P = (p_0, p_1, \cdots, p_{n-1})$ be a fan-shaped polygon. Let $\mathcal{P} = (X, \preccurlyeq) = VP(P)$ be the visibility partially ordered set of P.*

Then, $\text{HP}(P) = \text{Cover}(P) = \mu(\mathcal{P}) = \nu(\mathcal{P})$.

Proof This is implied by Lemma 9.24, Lemma 9.25, and Lemma 9.33. □

Theorem 9.35 *Let P be a fan-shaped polygon on n vertices, and $p_0 \in hub(P)$, then the maximum hidden point set of P can be computed in $O(n^2)$.*

Proof Let $\{\alpha_i\}_{i=1}^{n-2}$, $\{\beta_i\}_{i=1}^{n-2}$, $\{Q_i\}_{i=1}^{n-1}$, $\{l_{i,j}\}_{i<j}$, and $\{Y_i\}_{i=1}^{n-2}$ follow their definitions in Definition 9.27.

First, we argue that $\{\alpha_i\}_{i=1}^{n-2}$ and $\{\beta_i\}_{i=1}^{n-2}$ can be explicitly constructed in $O(n^2)$. By Lemma 4.10, $\forall i \in [n-1]$, Q_i can be computed in $O(n)$, and thus $\{Q_i\}_{i=1}^{n-1}$ is computable in $O(n^2)$.

Further, we can see that $\{l_{i,j}\}_{i<j}$ can also be in $O(n^2)$. $\forall i < j, I_P(p_i, p_j) = 0$, we can find the smallest integer $k = f(i,j) \in [i,j]$, such that $I_P(p_k, p_j) = 1$, and $\{f(i,j)\}_{i<j}$ can be computed in $O(n^2)$. Accordingly, $\{l_{i,j}\}_{i<j}$ can be computed in $O(n^2)$ since it takes $O(1)$ to check whether p_k is the only intermediate vertex in the geodesic path from p_i to p_j.

Therefore, we can compute both $\{Q_i\}_{i=1}^{n-1}$ and $\{l_{i,j}\}_{i<j}$ in $O(n^2)$, and thus $\{Y_i\}_{i=1}^{n-2}$, $\{\alpha_i\}_{i=1}^{n-2}$ and $\{\beta_i\}_{i=1}^{n-2}$ can be explicitly constructed in $O(n^2)$.

Let $S = \{\alpha_i\}_{i=1}^{n-2} \cup \{\beta_i\}_{i=1}^{n-2}$, by Lemma 9.33, there exists a hidden point set $H \subseteq S$ such that $|H| = \text{HP}(P)$. Consider S as the vertices of P, by Corollary 9.15, we can compute such maximum hidden point set H in $O(n^2)$, thus concluding the algorithm. □

Chapter 10

Conclusion

We conclude the thesis with the following conjectures and open problems.

- We conjecture that there exists constant $c > 1$, such that for any simple polygon P, $\text{HP}(P) \leq c\text{HV}(P)$.
- Conjecture 3.20: There exists constant $c > 0$, such that for any polygon P with h holes, $\text{HP}(P) \geq c\sqrt{h}$ stands.
- Conjecture 4.15: There exists constant $C > 0$, such that for any simple polygon P, let S be its set system of visible areas, we have $dim(S) < C$.
- We conjecture that the problem of finding the maximum hidden point set of a simple polygon is in NPO. In other words, given a simple polygon P, we can always find a maximum hidden point set H, which has polynomial bit size.
- Conjecture 6.12: There exist constants $\varepsilon > 0, c > 0$ such that for any simple polygon P on n vertices, $max\{\alpha(VG(P)), \omega(VG(P))\} \geq cn^{\varepsilon}$.
- We conjecture that the maximum hidden vertex set of a pseudotriangle can be computed in polynomial time.
- Let P be a simple polygon on n vertices and c of them are convex. We are interested in proposing a fixed parameter tractable algorithm $\mathcal{A}$, with time complexity $O(f(k)poly(n))$, which finds the maximum hidden vertex set of P or at least provides us with a non-trivial constant factor approximation.

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